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SEQUENTIAL CONJUGATE GRADIENT-RESTORATION ALGORITHM

FOR OPTIMAL CONTROL PROBLEMS

WITH NONDIFFERENTIAL CONSTRAINTS

AND GENERAL BOUNDARY CONDITIONS

by

A. K. WU and A. MIELE

RICE UNIVERSITY

1978

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20. ABSTRACT (Continue on reverse side if necessary and identify by block number)

This paper considers the numerical solution of the problem of minimizing a functional subject to differential constraints, nondifferential constraints, and general boundary conditions.

The approach taken is a sequence of two-phase cycles, composed of a conjugate gradient phase and a restoration phase. The conjugate gradient phase involves one iteration: the first variation of the functional is minimized, subject to the linearized constraints

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20. Abstract continued.

and a quadratic constraint imposed on the variations of the control, the parameter, and the missing components of the initial state. The restoration phase involves one or more iterations: in each iteration, the norm squared of the variations of the control, the parameter, and the missing components of the initial state is minimized, subject to the linearized constraints.

The algorithm has two principal properties: (i) it provides a sequence of suboptimal solutions; the functions obtained at the end of each cycle satisfy the constraints to a predetermined accuracy; and (ii) the value of the functional at the end of any complete conjugate gradient-restoration cycle is smaller than the value of the same functional at the beginning of that cycle.

The sequential conjugate gradient-restoration algorithm presented here differs from previous work, in that it is not required that the state vector be given at the initial point. Instead, the initial conditions can be absolutely general. Since the present algorithm is capable of handling general final conditions, it is suitable for the solution of optimal control problems with general boundary conditions. Its importance lies in the fact that many optimal control problems can be reduced to the format considered here. This includes (i) problems with state equality constraints, (ii) problems with control equality constraints, (iii) problems with state-derivative equality constraints, (iv) problems with state inequality constraints, (v) problems with control inequality constraints, (vi) problems with state-derivative inequality constraints, and (vii) minimax problems of optimal control.

Twelve numerical examples are presented in order to illustrate the performance of the algorithm. The numerical results show the feasibility as well as the convergence characteristics of the present algorithm.

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Sequential Conjugate Gradient-Restoration Algorithm

for Optimal Control Problems

with Nondifferential Constraints

and General Boundary Conditions[1]

by

A.K. WU[2] and A. MIELE[3]

Abstract. This paper considers the numerical solution of
the problem of minimizing a functional I subject to differen-
tial constraints, nondifferential constraints, and general
boundary conditions. It consists of finding the state $x(t)$,
the control $u(t)$, and the parameter π so that the functional
I is minimized while the constraints and the boundary condi-
tions are satisfied to a predetermined accuracy.

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University, Houston, Texas.

The approach taken is a sequence of two-phase cycles, composed of a conjugate gradient phase and a restoration phase. The conjugate gradient phase involves one iteration and is designed to decrease the value of the functional, while the constraints are satisfied to first order. During this iteration, the first variation of the functional is minimized, subject to linearized constraints. The minimization is performed over the class of variations of the control, the parameter, and the missing components of the initial state which are equidistant from some constant multiple of the corresponding variations of the previous conjugate gradient phase. The sequence of conjugate gradient phases generated by the algorithm is such that, for the special case of a quadratic functional subject to linear constraints, various orthogonality and conjugacy conditions hold. The restoration phase involves one or more iterations and is designed to force constraint satisfaction to a predetermined accuracy, while the norm squared of the variations of the control, the parameter, and the missing components of the initial state is minimized.

The principal property of the algorithm is that it produces a sequence of feasible suboptimal solutions: the functions obtained at the end of each cycle satisfy the constraints to a predetermined accuracy. Therefore, the values

of the functional I corresponding to any two elements of the sequence are comparable.

The stepsize of the conjugate gradient phase is determined by a one-dimensional search on the augmented functional J, while the stepsize of the restoration phase is obtained by a one-dimensional search on the constraint error P. The conjugate gradient stepsize and the restoration stepsize are chosen so that the restoration phase preserves the descent property of the conjugate gradient phase. Therefore, the value of the functional I at the end of any complete conjugate gradient-restoration cycle is smaller than the value of the same functional at the beginning of that cycle. Of course, restarting the algorithm might be occasionally necessary.

The sequential conjugate gradient-restoration algorithm presented here differs from that of Refs. 3 and 4, in that it is not required that the state vector be given at the initial point. Instead, the initial conditions can be absolutely general. In analogy with Refs. 3 and 4, the present algorithm is capable of handling general final conditions; therefore, it is suitable for the solution of optimal control problems with general boundary conditions. Its importance lies in the fact that many optimal control problems involve initial conditions of the type considered here.

Twelve numerical examples are presented in order to illustrate the performance of the algorithm. The numerical results show the feasibility as well as the convergence characteristics of the present algorithm.

<u>Key Words</u>. Optimal control, numerical methods, computing methods, gradient methods, gradient-restoration algorithms, sequential gradient-restoration algorithms, conjugate gradient-restoration algorithms, sequential conjugate gradient-restoration algorithms, nondifferential constraints, bounded control, bounded state, general boundary conditions.

1.   Introduction

Over the past several years, a successful family of al-
gorithms for the solution of optimal control problems involv-
ing differential constraints, nondifferential constraints,
and terminal constraints has been developed at Rice Universi-
ty by Miele and his associates (see Refs. 1-4). They are
known as sequential gradient-restoration algorithms and have
been designed for the solution of different classes of optimal
control problems. Some of these algorithms are of the
ordinary-gradient type (Refs. 1-2), while the rest are of the
conjugate-gradient type (Refs. 2-4). All of them have shown
to be robust and reliable; however, they all require the
state vector to be given at the initial point. Owing to the
fact that optimal control problems exist, which require satis-
faction of more general boundary conditions, the task of ex-
tending the aforementioned family of algorithms must be
undertaken, with these ideas in mind: (i) to retain the
robustness, reliability, and convergence characteristics of
the algorithms discussed in Refs. 1-4; (ii) to be able to
handle all of the optimal control problems treated in Refs.
1-4; and (iii) to have the additional capability of handling
optimal control problems with general boundary conditions.

In Ref. 5, an algorithm of the ordinary gradient type
was developed, extending the work of Ref. 1 to problems with

general boundary conditions. In this report, an algorithm of
the conjugate gradient type is developed, extending the work
of Refs. 3-4 to problems with general boundary conditions.

Specifically, the following optimal control problem is
considered: Minimize the functional I, which depends on the
n-vector state $x(t)$, the m-vector control $u(t)$, and the
p-vector parameter π. The state and the parameter are re-
quired to satisfy r scalar relations at the initial point and
q scalar relations at the final point. Along the interval of
integration, the state, the control, and the parameter are
required to satisfy n scalar differential equations and k
scalar nondifferential relations.

The approach taken is a sequence of two-phase cycles,
composed of a conjugate gradient phase and a restoration
phase. The conjugate gradient phase involves one iteration
and is designed to decrease the value of the functional, while
the constraints are satisfied to first order. During this iter-
ation, the first variation of the functional is minimized, sub-
ject to the linearized constraints. The minimization is per-
formed over the class of variations of the control, the para-
meter, and the missing components of the initial state which
are equidistant from some constant multiple of the corres-
ponding variations of the previous conjugate gradient phase.
The sequence of conjugate gradient phases generated by the

algorithm is such that, for the special case of a quadratic functional subject to linear constraints, various orthogonality and conjugacy conditions hold. The restoration phase involves one or more iterations and is designed to force constraint satisfaction to a predetermined accuracy, while the norm squared of the variations of the control, the parameter, and the missing components of the initial state is minimized.

The principal property of the algorithm is that it produces a sequence of feasible suboptimal solutions: the functions obtained at the end of each cycle satisfy the constraints to a predetermined accuracy. Therefore, the values of the functional I corresponding to any two elements of the sequence are comparable.

The stepsize of the conjugate gradient phase is determined by a one-dimensional search on the augmented functional J, while the stepsize of the restoration phase is obtained by a one-dimensional search on the constraint error P. The conjugate gradient stepsize and the restoration stepsize are chosen so that the restoration phase preserves the descent property of the conjugate gradient phase. Therefore, the value of the functional I at the end of any complete conjugate gradient-restoration cycle is smaller than the value of the same functional at the beginning of that cycle. Of course, restarting the algorithm might be occasionally necessary.

A time normalization is used in order to simplify the numerical computations. Specifically, the actual time θ is replaced by the normalized time $t = \theta/\tau$, which is defined in such a way that the initial time is $t = 0$ and the final time is $t = 1$. The actual final time τ, if it is free, is regarded as a component of the vector parameter π to be optimized. In this way, an optimal control problem with variable final time is converted into an optimal control problem with fixed final time.

2.   Statement of the Problem

   2.1.   Notation.   Let t denote the independent variable,
and let x(t), u(t), π denote the dependent variables.   The
time t is a scalar, the state x(t) is an n-vector, the con-
trol u(t) is an m-vector, and the parameter π is a p-vector.[4]

   The state x(t) is partitioned into vectors y(t) and z(t),
defined as follows: y(t) is an a-vector including those com-
ponents of the state that are prescribed at the initial point,
and z(t) is a b-vector including those components of the
state that are not prescribed at the initial point.   Clearly,
$a + b = n$.

   2.2.   Optimization Problem.   With the above notations,
the optimization problem can be stated as follows.   Minimize
the functional

$$I = \int_0^1 f(x,u,\pi,t)dt + [h(z,\pi)]_0 + [g(x,\pi)]_1 , \qquad (1)$$

with respect to the state x(t), the control u(t), and the
parameter π which satisfy the differential constraints

$$\dot{x} - \phi(x,u,\pi,t) = 0, \qquad 0 \le t \le 1, \qquad (2)$$

---

[4]All vectors are column vectors.

the nondifferential constraints

$$S(x,u,\pi,t) = 0, \qquad 0 \leq t \leq 1, \tag{3}$$

and the boundary conditions

$$y(0) = given, \tag{4}$$

$$[\omega(z,\pi)]_0 = 0, \tag{5}$$

$$[\psi(x,\pi)]_1 = 0. \tag{6}$$

In the above equations, the quantities I,f,h,g are scalar, the function ϕ is an n-vector, the function S is a k-vector, the function ω is a c-vector, and the function ψ is a q-vector. The number of initial conditions $r = a + c$ and the number of final conditions q must satisfy the following relation:

$$r + q \leq n_0 + n_1 + p_* \leq 2n + p. \tag{7}$$

In Ineq. (7), the symbol $p_* \leq p$ denotes the number of components of the parameter π present in the boundary conditions; the symbol $n_0 \leq n$ denotes the number of state variables present in the initial conditions; and the symbol $n_1 \leq n$ denotes the number of state variables present in the final conditions.

2.3. <u>First-Order Conditions</u>. From calculus of variations, it can be seen that the previous problem is one of the Bolza type, and it can be recast as that of minimizing the

agumented functional[5]

$$J = \int_0^1 [f + \lambda^T(\dot{x} - \phi) + \rho^T S] dt + (h + \sigma^T \omega)_0 + (g + \mu^T \psi)_1$$

$$= \int_0^1 (f - \lambda^T \phi + \rho^T S - \dot{\lambda}^T x) dt + (-\lambda^T x + h + \sigma^T \omega)_0 + (\lambda^T x + g + \mu^T \psi)_1, \quad (8)$$

subject to (2)-(6). Here, the n-vector $\lambda(t)$ is a variable Lagrange multiplier, the k-vector $\rho(t)$ is a variable Lagrange multiplier, the c-vector σ is a constant Lagrange multiplier, and the q-vector μ is a constant Lagrange multiplier.

The functions $x(t)$, $u(t)$, π and the multipliers $\lambda(t)$, $\rho(t)$, σ, μ solving the previous problem must satisfy the feasibility equations (2)-(6) and the following optimality conditions:

$$\dot{\lambda} - f_x + \phi_x \lambda - S_x \rho = 0, \qquad 0 \le t \le 1, \qquad (9)$$

$$f_u - \phi_u \lambda + S_u \rho = 0, \qquad 0 \le t \le 1, \qquad (10)$$

$$\int_0^1 (f_\pi - \phi_\pi \lambda + S_\pi \rho) dt + (h_\pi + \omega_\pi \sigma)_0 + (g_\pi + \psi_\pi \mu)_1 = 0, \qquad (11)$$

$$(-\zeta + h_z + \omega_z \sigma)_0 = 0, \qquad (12)$$

$$(\lambda + g_x + \psi_x \mu)_1 = 0. \qquad (13)$$

---

[5] The second form of Eq. (8) arises after the customary integration by parts is performed.

2.4. <u>Remark</u>. Just as the state vector x(t) is parti-
tioned into an a-vector y(t) and a b-vector z(t), the multi-
plier vector λ(t) is partitioned into an a-vector η(t) and a
b-vector ζ(t), having the following meaning: η(t) is associa-
ted with y(t), and ζ(t) is associated with z(t). With
reference to Eq. (12), ζ(0) denotes the portion of the initial
Lagrange multiplier λ(0) which is associated with z(0), the
portion of the initial state vector x(0) which is not pres-
cribed.

2.5. <u>Approximate Methods</u>. Since in general the differ-
ential system (2)-(6) and .(9)-(13) is nonlinear, approximate
methods are employed to find a solution iteratively. In this
connection, let the norm squared of a vector ν be defined by

$$N(\nu) = \nu^T \nu . \tag{14}$$

Then, the constraint error P can be written as[6]

$$P = \int_0^1 N(\dot{x} - \phi)\,dt + \int_0^1 N(S)\,dt + N(\omega)_0 + N(\psi)_1 , \tag{15}$$

and the error in the optimality conditions Q is given by

---

[6] In Eq. (15), it is assumed that the initial condition (4) is
satisfied.

$$Q = \int_0^1 N(\dot{\lambda} - f_x + \phi_x \lambda - S_x \rho)dt + \int_0^1 N(f_u - \phi_u \lambda + S_u \rho)dt$$

$$+ N\left[\int_0^1 (f_\pi - \phi_\pi \lambda + S_\pi \rho)dt + (h_\pi + \omega_\pi \sigma)_0 + (g_\pi + \psi_\pi \mu)_1\right]$$

$$+ N(-\zeta + h_z + \omega_z \sigma)_0 + N(\lambda + g_x + \psi_x \mu)_1 . \qquad (16)$$

For the exact optimal solution, one must have

$$P = 0, \qquad Q = 0. \qquad (17)$$

For an approximation to the optimal solution, the following
relations are to be satisfied:

$$P \leq \epsilon_1 , \qquad Q \leq \epsilon_2 , \qquad (18)$$

where ϵ_1 and ϵ_2 are small, preselected numbers.

3.   Description of the Algorithm

The technique employed is characterized by a sequence of
two-phase cycles, composed of a conjugate gradient phase and
a restoration phase. These phases are described below.

The conjugate gradient phase is started only when the
constraint error P satisfies Ineq. (18-1). It involves a
single iteration, which is designed to decrease the value of
the functional I or the augmented functional J, while the
constraints are satisfied to first order. During this itera-
tion, the first variation of the functional I is minimized,
subject to the linearized constraints. The minimization is
performed over the class of variations of the control $u(t)$ and
the parameters π and $z(0)$ which are equidistant from some con-
stant multiple of the corresponding variations of the previous
conjugate gradient phase.[7]

The restoration phase is started only when the constraint
error P violates Ineq. (18-1). The restoration phase involves
one or more iterations. In each restorative iteration, the
objective is to reduce the constraint error P, while the
constraints are satisfied to first order and the norm squared

---

[7]The sequence of conjugate gradient phases generated by the
algorithm is such that, for the special case of a quadratic
functional subject to linear constraints, various orthogonal-
ity and conjugacy conditions hold (see Section 6).

of the variations of the control u(t) and the parameters π
and z(0) is minimized. The restoration phase is terminated
whenever Ineq. (18-1) is satisfied.

A complete conjugate gradient-restoration cycle is de-
signed so that the value of the functional I decreases,while
the constraints are satisfied to the accuracy (18-1) both at
the beginning and at the end of the cycle. Finally, the al-
gorithm as a whole is terminated whenever Ineqs. (18) are
satisfied simultaneously.

3.1. Remark. During each conjugate gradient iteration
or restorative iteration, use of nonlinear equations must be
avoided. Therefore, the exact feasibility equations (2)-(6)
are replaced by their corresponding linearized approximations.
These linearized approximations do not include forcing terms
in the conjugate gradient phase, while they do include forcing
terms in the restoration phase.

3.2. Notation. For any iteration of the conjugate
gradient phase or the restoration phase, the following termi-
nology is adopted: $x(t)$, $u(t)$, π denote the nominal functions;
$\tilde{x}(t)$, $\tilde{u}(t)$, $\tilde{\pi}$ denote the varied functions; and $\Delta x(t)$, $\Delta u(t)$,
$\Delta\pi$ denote the displacements leading from the nominal functions
to the varied functions. These quantities satisfy the relations

$$\tilde{x}(t) = x(t) + \Delta x(t), \qquad \tilde{u}(t) = u(t) + \Delta u(t), \qquad \tilde{\pi} = \pi + \Delta\pi. \quad (19)$$

Because the vector x is partitioned into y and z, Eq. (19-1) implies that

$$\tilde{y}(t) = y(t) + \Delta y(t), \qquad \tilde{z}(t) = z(t) + \Delta z(t). \qquad (20)$$

Let α be a positive number representing the stepsize (either the conjugate gradient stepsize or the restoration stepsize). Then, we define the displacements per unit stepsize as follows:

$$A(t) = \Delta x(t)/\alpha, \qquad B(t) = \Delta u(t)/\alpha, \qquad C = \Delta \pi/\alpha. \qquad (21)$$

The vector A is partitioned into vectors D and E associated with y and z, respectively. Therefore, Eq. (21-1) implies that

$$D(t) = \Delta y(t)/\alpha, \qquad E(t) = \Delta z(t)/\alpha. \qquad (22)$$

Upon combining (19)-(20) with (21)-(22), we see that

$$\tilde{x}(t) = x(t) + \alpha A(t), \qquad \tilde{u}(t) = u(t) + \alpha B(t), \qquad \tilde{\pi} = \pi + \alpha C, \qquad (23)$$

$$\tilde{y}(t) = y(t) + \alpha D(t), \qquad \tilde{z}(t) = z(t) + \alpha E(t). \qquad (24)$$

3.3. <u>Desired Properties</u>. The functions $\Delta x(t)$, $\Delta u(t)$, $\Delta \pi$ must be determined so as to produce some desirable effect at every iteration, namely, the decrease of the functionals I, and/or J, and/or P. Thus, the following descent properties are required:

$$\tilde{I} < I, \quad \text{and/or} \quad \tilde{J} < J, \quad \text{and/or} \quad \tilde{P} < P, \quad (25)$$

where I, J, P are associated with the nominal functions and $\tilde{I}, \tilde{J}, \tilde{P}$ are associated with the varied functions. In turn, the functions A(t), B(t), C are chosen so that

$$\delta I < 0, \quad \text{and/or} \quad \delta J < 0, \quad \text{and/or} \quad \delta P < 0, \quad (26)$$

where the symbol $\delta(...)$ denotes the first variation. Then, by choosing the stepsize α sufficiently small, the satisfaction of relations (25) is guaranteed. Ineqs. (25-1), (25-2) and (26-1), (26-2) characterize the conjugate gradient phase, while Ineqs. (25-3) and (26-3) characterize the restoration phase.

3.4. <u>First Variations</u>. Next, we give the expressions for the first variations of the functionals I, J, P; after simple manipulations, omitted for the sake of brevity, they take the form[8,9]

$$\delta I/\alpha = \int_0^1 (f_x^T A + f_u^T B + f_\pi^T C)\,dt + (h_z^T E + h_\pi^T C)_0 + (g_x^T A + g_\pi^T C)_1, \quad (27)$$

---

[8] Implicit in Eqs. (27)-(29) is the assumption D(0) = 0.

[9] The first variation of the augmented functional J is computed by varying the functions x(t), u(t), π, while holding the multipliers $\lambda(t)$, $\rho(t)$, σ, μ unchanged.

and

$$\delta J/\alpha = \int_0^1 (-\dot{\lambda} + f_x - \phi_x\lambda + S_x\rho)^T A \, dt + \int_0^1 (f_u - \phi_u\lambda + S_u\rho)^T B \, dt$$

$$+ \left[\int_0^1 (f_\pi - \phi_\pi\lambda + S_\pi\rho) \, dt + (h_\pi + \omega_\pi\sigma)_0 + (g_\pi + \psi_\pi\mu)_1 \right]^T C$$

$$+ \left[(-\zeta + h_z + \omega_z\sigma)^T E \right]_0 + \left[(\lambda + g_x + \psi_x\mu)^T A \right]_1 \quad , \tag{28}$$

and

$$\delta P/2\alpha = \int_0^1 (\dot{x}-\phi)^T (\dot{A} - \phi_x^T A - \phi_u^T B - \phi_\pi^T C) \, dt + \int_0^1 S^T (S_x^T A + S_u^T B + S_\pi^T C) \, dt$$

$$+ \left[\omega^T (\omega_z^T E + \omega_\pi^T C) \right]_0 \quad + \quad \left[\psi^T (\psi_x^T A + \psi_\pi^T C) \right]_1 \; . \tag{29}$$

3.5. <u>Remark</u>. For the purpose of this report, Eqs. (27)-(29) must be completed by one of the following relations:

$$K/\alpha^2 = \int_0^1 B^T B \, dt + C^T C + (E^T E)_0 , \tag{30}$$

or

$$K/\alpha^2 = \int_0^1 (B-\gamma\hat{B})^T (B-\gamma\hat{B}) \, dt + (C-\gamma\hat{C})^T (C-\gamma\hat{C}) + \left[(E-\gamma\hat{E})^T (E-\gamma\hat{E}) \right]_0 , \tag{31}$$

which constitute a measure of the overall change of the control $u(t)$ and the parameters π and $z(0)$. Equation (30) is employed in the restoration phase, and Eq. (31) is employed in the conjugate gradient phase. In Eq. (31), the functions $A(t)$, $B(t)$, C pertain to the present conjugate gradient phase, and the functions $\hat{A}(t)$, $\hat{B}(t)$, $\hat{C}$ pertain to the previous conjugate gradient phase. The symbol γ denotes a scalar, nonnegative quantity, called the directional coefficient. Its specification is discussed in Section 6.

4.   Restoration Phase

4.1.   Linearized Equations.   Let $x(t)$, $u(t)$, π denote nominal functions satisfying (4), but not necessarily (2)-(3) and (5)-(6).   To first order, the perturbations per unit stepsize $A(t)$, $B(t)$, C must satisfy the linearized constraint equations

$$\dot{A} - \phi_x^T A - \phi_u^T B - \phi_\pi^T C + (\dot{x} - \phi) = 0, \qquad 0 \le t \le 1, \qquad (32)$$

$$S_x^T A + S_u^T B + S_\pi^T C + S = 0, \qquad 0 \le t \le 1, \qquad (33)$$

$$D(0) = 0, \qquad (34)$$

$$(\omega_z^T E + \omega_\pi^T C + \omega)_0 = 0, \qquad (35)$$

$$(\psi_x^T A + \psi_\pi^T C + \psi)_1 = 0. \qquad (36)$$

4.2.   Descent Property.   The linearized equations (32)-(36) admit an infinite number of solutions, each of which is characterized by a descent property in the constraint error P.   This descent property can be seen by combining (29) with (32)-(36):   the first variation of P becomes

$$\delta P = -2\alpha P. \qquad (37)$$

Since $P > 0$, Eq. (37) shows that $\delta P < 0$; hence, for α suffi-
ciently small, a decrease in the constraint error P is guar-
anteed.

4.3.   <u>Special Variations</u>.   Among the infinite number of
solutions of Eqs. (32)-(36), the one that minimizes the
functional (30) is selected. Thus, we seek the minimum of
the quadratic functional (30), with respect to the functions
$A(t)$, $B(t)$, C which satisfy the linearized constraints
(32)-(36).

By applying standard techniques of optimal control theory
or calculus of variations, the following optimality conditions
are obtained:

$$B = \phi_u \lambda - S_u \rho, \qquad 0 \le t \le 1, \qquad (38)$$

$$C = \int_0^1 (\phi_\pi \lambda - S_\pi \rho)\, dt - (\omega_\pi \sigma)_0 - (\psi_\pi \mu)_1, \qquad (39)$$

$$E(0) = (\zeta - \omega_z \sigma)_0, \qquad (40)$$

$$\dot{\lambda} + \phi_x \lambda - S_x \rho = 0, \qquad 0 \le t \le 1, \qquad (41)$$

$$(\lambda + \psi_x \mu)_1 = 0. \qquad (42)$$

Summarizing, we seek functions $A(t)$, $B(t)$, C and multipliers

$\lambda(t)$, $\rho(t)$, σ, μ which satisfy the linearized constraints
(32)-(36) and the optimality conditions (38)-(42).

4.4. Linear, Two-Point Boundary-Value Problem. The
technique used to solve the LTP-BVP (32)-(36) and (38)-(42)
is a forward integration scheme in combination with the method
of particular solutions (Refs. 6-8). The technique requires
the execution of $n+p+1$ independent sweeps of the differen-
tial system (32)-(36) and (38)-(42), each characterized by a
different value of the $(n+p)$-vector w, whose components are
the n components of the initial multiplier $\lambda(0)$ and the p
components of the parameter C.

The generic sweep is started by assigning particular
values to the components of w, that is, the components of the
vectors $\lambda(0)$ and C. With $\lambda(0)$ known, $\zeta(0)$ is known. There-
fore, Eqs. (35) and (40) constitute a system of b+c linear
relations in which the unknowns are the b+c components of
the vectors E(0) and σ. For this system to have a unique
solution, the following disequation must hold:[10]

$$\det \left[\omega_z^T \omega_z \right]_0 \neq 0 . \tag{43}$$

[10] Disequation (43) is obtained from (35) and (40) after elimin-
ation of E(0). The resulting linear equation in σ admits a
unique solution providing (43) is satisfied.

With E(0) known and because of (34), the vector A(0) is known.
Then, A(t) and λ(t) together with B(t) and ρ(t) are obtained
by forward integration of (32) and (41), subject to (33) and
(38).   Note that, at each time station t, $0 \leq t \leq 1$, Eqs. (33)
and (38) constitute a system of $m+k$ linear relations in which
the unknowns are the $m+k$ components of the vectors B(t) and
ρ(t). For this system to have a unique solution, the follow-
ing disequation must hold:[11]

$$\det \left[S_u^T S_u \right] \neq 0, \qquad 0 \leq t \leq 1. \qquad (44)$$

As a result of the procedure, the sweep is completed: for the
arbitrary value assigned to w, it leads to the satisfaction of
all of the equations of the system (32)-(36) and (38)-(42),
except Eqs. (36), (39), (42).

In order to satisfy Eqs. (36), (39), (42) and because
the system (32)-(36) and (38)-(42) is nonhomogeneous, $n+p+1$
independent sweeps must be executed employing $n+p+1$ differ-
ent vectors w_i, $i=1,\ldots,n+p+1$. The first $n+p$ sweeps are
performed by choosing the vectors $w_1,\ldots\ldots,w_{n+p}$ to be the
columns of the identity matrix of order $n+p$. The last sweep

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[11]Disequation (44) is obtained from (33) and (38) after elim-
ination of B(t). The resulting linear equation in ρ(t)
admits a unique solution providing (44) is satisfied.

is executed by choosing w_{n+p+1} to be the null vector. As
a result, one generates the functions and multipliers

$$A_i(t), B_i(t), C_i, \lambda_i(t), \rho_i(t), \sigma_i, \ i = 1, \ldots, n+p+1 . \quad (45)$$

Now, we introduce the $n+p+1$ undetermined, scalar con-
stants k_i and form the linear combinations

$$A(t) = \Sigma k_i A_i(t), \qquad B(t) = \Sigma k_i B_i(t), \qquad C = \Sigma k_i C_i , \quad (46)$$

$$\lambda(t) = \Sigma k_i \lambda_i(t), \qquad \rho(t) = \Sigma k_i \rho_i(t), \qquad \sigma = \Sigma k_i \sigma_i , \quad (47)$$

where the summations are taken over the index i. The $n+p+1$
coefficients k_i and the q components of the multiplier μ are
obtained by forcing the linear combinations (46)-(47) to sat-
isfy Eqs. (36), (39), (42), together with the normalization
condition (Ref. 6)

$$\Sigma k_i = 1 . \quad (48)$$

Once the constants k_i are known, the solution of the LTP-BVP
(32)-(36) and (38)-(42) is given by (46)-(47).

    4.5. _Restoration Stepsize_. With the functions A(t),
B(t), C known, the one-parameter family of varied functions
(23) can be formed. For this one-parameter family, the con-
straint error (15) becomes a function of the form

$$\tilde{P} = \tilde{P}(\alpha) . \tag{49}$$

Then, the stepsize α must be selected so that the following
relations are satisfied:

$$\tilde{P}(\alpha) < \tilde{P}(0), \qquad \tilde{\tau}(\alpha) > 0. \tag{50}$$

Satisfaction of Ineq. (50-1) is possible because of the des-
cent property of the restoration phase. Ineq. (50-2) is
required for problems with free final time.

In order to achieve satisfaction of (50), a bisection
process is applied to the restoration stepsize α, starting
from the reference stepsize $\alpha_o = 1$. This reference stepsize
has the property of yielding one-step restoration for the case
where the constraints (2)-(6) are linear.

4.6. Iterative Procedure for the Restoration Phase.
The descent property (37) of the restoration phase guarantees
satisfaction of Ineq. (50-1) at the end of any iteration, but
not satisfaction of Ineq. (18-1). Therefore, the restoration
algorithm must be employed iteratively until Ineq. (18-1) is
satisfied. At this point, the restoration phase is termina-
ted.

5.   Conjugate Gradient Phase: General Case

    5.1. Linearized Equations. Suppose that nominal func-
tions $x(t)$, $u(t)$, π are available, which satisfy (2)-(6). To
first order, the perturbations per unit stepsize $A(t)$, $B(t)$,
C must satisfy the linearized constraint equations

$$\dot{A} - \phi_x^T A - \phi_u^T B - \phi_\pi^T C = 0, \qquad 0 \le t \le 1, \tag{51}$$

$$S_x^T A + S_u^T B + S_\pi^T C = 0, \qquad 0 \le t \le 1, \tag{52}$$

$$D(0) = 0, \tag{53}$$

$$(\omega_z^T E + \omega_\pi^T C)_0 = 0, \tag{54}$$

$$(\psi_x^T A + \psi_\pi^T C)_1 = 0. \tag{55}$$

    5.2. Special Variations. Among the infinite number of
solutions of Eqs. (51)-(55), the one that minimizes the func-
tional (27) is selected. Thus, we seek the minimum of the
linear functional (27), with respect to the functions $A(t)$,
$B(t)$, C which satisfy the linearized constraints (51)-(55)
and the quadratic isoperimetric constraint (31).

    By applying standard techniques of optimal control theory
or calculus of variations, the following optimality conditions

are obtained:[12]

$$B = \gamma\hat{B} - (f_u - \phi_u\lambda + S_u\rho), \qquad 0 \le t \le 1, \tag{56}$$

$$C = \gamma\hat{C} - \left[\int_0^1 (f_\pi - \phi_\pi\lambda + S_\pi\rho)dt + (h_\pi + \omega_\pi\sigma)_0 + (g_\pi + \psi_\pi\mu)_1 \right], \tag{57}$$

$$E(0) = \gamma\hat{E}(0) - (-\zeta + h_z + \omega_z\sigma)_0, \tag{58}$$

$$\dot{\lambda} - f_x + \phi_x\lambda - S_x\rho = 0, \qquad 0 \le t \le 1, \tag{59}$$

$$(\lambda + g_x + \psi_x\mu)_1 = 0. \tag{60}$$

Summarizing, for a given value of the directional coefficient γ, we seek functions $A(t)$, $B(t)$, C and multipliers $\lambda(t)$, $\rho(t)$, σ, μ which satisfy the linearized constraints (51)-(55) and the optimality conditions (56)-(60).

5.3. <u>Isoperimetric Constraint.</u> In the light of (56)-(60), the error in the optimality conditions (16) reduces to

$$Q = \int_0^1 (B-\gamma\hat{B})^T (B-\gamma\hat{B})dt + (C-\gamma\hat{C})^T (C-\gamma\hat{C}) + \left[(E-\gamma\hat{E})^T (E-\gamma\hat{E}) \right]_0. \tag{61}$$

---

[12]In Eqs. (56)-(60), the multiplier ν associated with the isoperimetric constraint (31) is set at the level $2\nu = 1$.

Consequently, the following relation ties the isoperimetric
constant, the stepsize, and the error in the optimality con-
ditions:

$$K = \alpha^2 Q .$$ (62)

Clearly, to assign values to the isoperimetric constant
is the same as assigning values to the stepsize. However,
there is no clear-cut way of determining a priori convenient
values for the isoperimetric constant. Therefore, the imple-
mentation of the conjugate gradient algorithm becomes simpler
if one avoids evaluating α in terms of K through (62) and
assigns values to α directly.

5.4. <u>Descent Property</u>. When the variations defined by
(51)-(60) are employed, the first variation of the augmented
functional (28) becomes

$$\delta J = -\alpha (Q + \gamma Z) ,$$ (63)

where Q is given by (61) and

$$Z = \int_0^1 (B - \gamma \hat{B})^T \hat{B} dt + (C - \gamma \hat{C})^T \hat{C} + \left[(E - \gamma \hat{E})^T \hat{E} \right]_0 .$$ (64)

For the first iteration of the conjugate gradient phase,
one sets

$$\gamma = 0 ,$$ (65)

with the implication that

$$\delta J = -\alpha Q . \qquad (66)$$

Since $Q > 0$, Eq. (66) shows that $\delta J < 0$. Hence, for α suffi-
ciently small, it is guaranteed that the augmented functional
J decreases.

For subsequent iterations, one sets $\gamma \neq 0$. More specifi-
cally, the directional coefficient must be such that

$$\gamma > 0, \qquad (67)$$

and its proper value is discussed in Section 6. At any rate,
Eq. (63) shows that $\delta J < 0$ providing

$$Q + \gamma Z > 0 . \qquad (68)$$

Hence, for α sufficiently small, it is guaranteed that the
augmented functional J decreases as long as Ineq. (68) is
satisfied. If Ineq. (68) is violated, the descent property
on J no longer holds, and the conjugate gradient phase must
be restarted by resetting the directional coefficient γ at the
level (65).

5.5. <u>Linear, Two-Point Boundary-Value Problem</u>. For a
given value of the directional coefficient γ, the technique
used to solve the LTP-BVP (51)-(60), associated with the con-
jugate gradient phase, is analogous to that described for the

restoration phase (see Section 4.4); hence, it is not re-
peated, for the sake of brevity.

    5.6. <u>General Solution</u>. Next, assume that two particu-
lar values are given to the directional coefficient γ, for
instance,

$$\gamma_* = 0 \qquad \text{and} \qquad \gamma_{**} = 1 . \tag{69}$$

Denote by

$$A_*(t), \ B_*(t), \ C_*, \ \lambda_*(t), \ \rho_*(t), \ \sigma_*, \mu_* \tag{70}$$

and

$$A_{**}(t), \ B_{**}(t), \ C_{**}, \lambda_{**}(t), \ \rho_{**}(t), \ \sigma_{**}, \mu_{**} \tag{71}$$

the particular solutions of the LTP-BVP (51)-(60) correspond-
ing to (69-1) and (69-2), respectively. Simple manipulations,
omitted for the sake of brevity, show that the general solu-
tion of (51)-(60), valid for any value of the directional
coefficient γ, can be written as

$$A(t) = A_*(t) + \gamma [A_{**}(t) - A_*(t)] , \tag{72-1}$$

$$B(t) = B_*(t) + \gamma [B_{**}(t) - B_*(t)], \tag{72-2}$$

$$C = C_* + \gamma (C_{**} - C_*), \tag{72-3}$$

and

$$\lambda(t) = \lambda_*(t) + \gamma[\lambda_{**}(t) - \lambda_*(t)], \qquad (73\text{-}1)$$

$$\rho(t) = \rho_*(t) + \gamma[\rho_{**}(t) - \rho_*(t)], \qquad (73\text{-}2)$$

$$\sigma = \sigma_* + \gamma(\sigma_{**} - \sigma_*), \qquad (73\text{-}3)$$

$$\mu = \mu_* + \gamma(\mu_{**} - \mu_*). \qquad (73\text{-}4)$$

As a conclusion, the general solution of (51)-(60) requires
that two sets of $n + p + 1$ sweeps be executed, one leading to
the particular solution (70) and one leading to the particu-
lar solution (71).

5.7. <u>Stepsize and Directional Coefficient</u>. With the
functions $A(t)$, $B(t)$, C known, the following two-parameter
family of varied functions can be formed:

$$\tilde{x}(t) = x(t) + \alpha\left\{A_*(t) + \gamma[A_{**}(t) - A_*(t)]\right\}, \qquad (74\text{-}1)$$

$$\tilde{u}(t) = u(t) + \alpha\left\{B_*(t) + \gamma[B_{**}(t) - B_*(t)]\right\}, \qquad (74\text{-}2)$$

$$\tilde{\pi} = \pi + \alpha[C_* + \gamma(C_{**} - C_*)]. \qquad (74\text{-}3)$$

On the other hand, the multipliers $\lambda(t), \rho(t), \sigma, \mu$ form the
one-parameter family (73). Upon using (73) and (74), we see
that the augmented functional (8) takes the form

$$\tilde{J} = \tilde{J}(\alpha, \gamma) \ . \tag{75}$$

Therefore, the optimum values of α and γ satisfy the relations

$$\tilde{J}_\alpha(\alpha, \gamma) = 0, \qquad \tilde{J}_\gamma(\alpha, \gamma) = 0 \ . \tag{76}$$

Since the simultaneous determination of α and γ might be be expensive computationally, we proceed in a different way. First, we determine an approximate value of the directional coefficient γ, based on the consideration of the linear-quadratic model (Section 6). Once γ is known, the two-parameter family (75) reduces to the one-parameter family

$$\tilde{J} = \tilde{J}(\alpha) \ . \tag{77}$$

Then, the optimum stepsize α satisfies the relation

$$\tilde{J}_\alpha(\alpha) = 0 \ , \tag{78}$$

whose numerical solution can be obtained using quadratic interpolation or cubic interpolation (Ref. 9).

6.   Conjugate Gradient Phase: Linear-Quadratic Case

In the previous section, we analyzed the conjugate gradient phase in general, regardless of the analytical form of the functional (1) and the constraints (2)-(6). In this section, we consider the linear-quadratic case, that is, the case where the functional (1) is quadratic and the constraints (2)-(6) are linear.

6.1. General Solution. Under the assumption of linear constraints, it can be verified that the particular solutions (70) and (71) satisfy the relations

$$A_{**}(t)-A_{*}(t)=\hat{A}(t), \qquad B_{**}(t)-B_{*}(t)=\hat{B}(t), \qquad C_{**}-C_{*}=\hat{C}, \quad (79)$$

$$\lambda_{**}(t)-\lambda_{*}(t)=0, \quad \rho_{**}(t)-\rho_{*}(t)=0, \quad \sigma_{**}-\sigma_{*}=0, \quad \mu_{**}-\mu_{*}=0 . \quad (80)$$

As a consequence, Eqs. (72)-(73) reduce to

$$A(t) = A_{*}(t) + \gamma\hat{A}(t), \qquad B(t)=B_{*}(t) + \gamma\hat{B}(t), \qquad C = C_{*}+ \gamma\hat{C}, \quad (81)$$

$$\lambda(t)=\lambda_{*}(t), \qquad \rho(t)=\rho_{*}(t), \qquad \sigma=\sigma_{*}, \qquad \mu=\mu_{*} . \qquad (82)$$

This means that the general solution of the LTP-BVP (51)-(60) can be obtained by executing only one set of $n+p+1$ sweeps, namely, the set of sweeps leading to the solution (70). By the way, this is the solution corresponding to (69-1), namely, the solution associated with the ordinary gradient phase of Ref. 5.

6.2. Local Orthogonality Conditions. Under the assump-
tion of linear constraints, the two-parameter family (74)
simplifies to

$$\tilde{x}(t) = x(t) + \alpha[A_*(t) + \gamma\hat{A}(t)],\qquad\qquad (83\text{-}1)$$

$$\tilde{u}(t) = u(t) + \alpha[B_*(t) + \gamma\hat{B}(t)],\qquad\qquad (83\text{-}2)$$

$$\tilde{\pi} = \pi + \alpha(C_* + \gamma\hat{C}).\qquad\qquad (83\text{-}3)$$

On the other hand, the multipliers $\lambda(t)$, $\rho(t)$, σ, μ are given
by Eqs. (82). Upon using (82) and (83), we see that the
augmented functional (8) still takes the form (75). Hence,
the optimum values of α and γ still satisfy the relations (76).

   After laborious manipulations, omitted for the sake of
brevity, Eqs. (76) lead to the following local orthogonality
conditions:

$$\int_0^1 \tilde{B}_*^T B\,dt + \tilde{C}_*^T C + (\tilde{E}_*^T E)_0 = 0,\qquad\qquad (84\text{-}1)$$

$$\int_0^1 \tilde{B}_*^T \hat{B}\,dt + \tilde{C}_*^T \hat{C} + (\tilde{E}_*^T \hat{E})_0 = 0,\qquad\qquad (84\text{-}2)$$

with the implication that

$$\int_0^1 \tilde{B}_*^T B_*\,dt + \tilde{C}_*^T C_* + (\tilde{E}_*^T E_*)_0 = 0.\qquad\qquad (84\text{-}3)$$

Here, the adjective _local_ is employed to mean that Eqs. (84)
involve vectors B(t), C, E(0) which are solutions of (51)-(60)
computed for the present iteration and the previous iteration;
they also involve vectors $B_*(t)$, C_*, $E_*(0)$ which are solutions
of (51)-(60) for $\gamma = 0$ computed for the present iteration and
the next iteration.

    6.3. Local Conjugacy Condition. Let w(t) and M(t)
denote the vectors

$$w(t) = \begin{bmatrix} x(t) \\ u(t) \\ \pi \end{bmatrix}, \qquad M(t) = \begin{bmatrix} A(t) \\ B(t) \\ C \end{bmatrix} . \qquad (85)$$

Let f_{ww}, g_{ww}, h_{ww} denote the Hessian matrices of the functions
f,g,h with respect to the vector w. With this notation, and
under the assumption of linear constraints and quadratic
functional, Eqs. (76) lead to the following local conjugacy
condition:

$$\int_0^1 M^T f_{ww} \hat{M} dt + (M^T h_{ww} \hat{M})_0 + (M^T g_{ww} \hat{M})_1 = 0. \qquad (86)$$

Here, the adjective _local_ is employed to mean that Eq. (86)
involves vectors M(t), that is, vectors A(t), B(t), C,which
are solutions of (51)-(60) computed for the present iteration
and the previous iteration.

6.4.   <u>Stepsize and Directional Coefficient</u>.   After
observing that

$$M(t) = M_*(t) + \gamma \hat{M}(t), \tag{87}$$

and after accounting for Eqs. (84) and (86), it can be shown
that the optimal values of γ and α are given by

$$\gamma = Q/\hat{Q} \quad , \quad \alpha = Q/R , \tag{88}$$

where

$$Q = \int_0^1 B_*^T \dot{B}_* dt + C_*^T C_* + (E_*^T E_*)_0 , \tag{89-1}$$

$$\hat{Q} = \int_0^1 \hat{B}_*^T \hat{B}_* dt + \hat{C}_*^T \hat{C}_* + (\hat{E}_*^T \hat{E}_*)_0 , \tag{89-2}$$

$$R = \int_0^1 M^T f_{ww} M dt + (M^T h_{ww} M)_0 + (M^T g_{ww} M)_1 . \tag{89-3}$$

Clearly, the optimal directional coefficient γ is the
ratio of the error in the optimality conditions Q for the
present conjugate gradient iteration to the error in the
optimality conditions $\hat{Q}$ for the previous conjugate gradient
iteration. These quantities are known, since they involve
vectors $B_*(t)$, C_*, $E_*(0)$ which are solutions of (51)-(60) for
$\gamma = 0$ computed for the present iteration and the previous
 iteration

With γ known, the vector $M(t)$ is computed with (87) and
the scalar quantity R is computed with (89-3). Then, the
optimal stepsize α is evaluated with (88-2). Clearly, the
optimal stepsize α is the ratio of the error in the optimal-
ity conditions Q to the scalar quantity R, which constitutes
a measure of the curvature of the functional (1). Both Q
and R are computed employing quantities pertaining to the
present conjugate gradient iteration.

6.5. Descent Property. Under the assumption of linear
constraints, Eq. (64) simplifies to

$$Z = \int_0^1 B_*^T \hat{B} dt + C_*^T \hat{C} + (E_*^T \hat{E})_0 . \tag{90}$$

Because of the local orthogonality condition (84-1) written
for the previous iteration, Eq. (90) yields

$$Z = 0. \tag{91}$$

As a consequence, Eq. (63) reduces to

$$\delta J = -\alpha Q, \tag{92}$$

where the error in the optimality conditions Q is given by Eq.
(89-1). Equation (92) holds for any conjugate gradient iter-
ation and shows that, since $Q > 0$, we have $\delta J < 0$. Hence, for
α sufficiently small, it is guaranteed that the augmented

functional J decreases. In conclusion, for the linear-quadratic case, the restart procedure mentioned in Section 5.4 never occurs. This means that the directional coefficient γ is set at the level (65) only for the first conjugate gradient iteration.

6.6. <u>General Orthogonality and Conjugacy Conditions.</u> Now, assume that the algorithm described by Eqs. (51)-(60) and (83) is employed, starting with some feasible nominal functions. Further, assume that the directional coefficient γ is set at the level (65) for the first conjugate gradient iteration and at the level (88-1) for any subsequent conjugate gradient iteration. Under these assumptions and for the linear-quadratic case, one can generalize the local orthogonality conditions (84) and the local conjugacy condition (86) as follows:

$$\int_0^1 B_\star^T B_p dt + C_\star^T C_p + (E_\star^T E_p)_0 = 0, \tag{93-1}$$

$$\int_0^1 B_\star^T B_{\star p} dt + C_\star^T C_{\star p} + (E_\star^T E_{\star p})_0 = 0, \tag{93-2}$$

and

$$\int_0^1 M^T f_{ww} M_p dt + (M^T h_{ww} M_p)_0 + (M^T g_{ww} M_p)_1 = 0, \tag{93-3}$$

where the subscript p denotes any conjugate gradient itera-
tion preceding the present conjugate gradient iteration.
While these equations do not guarantee convergence in a finite
number of steps, they do guarantee that the algorithm gener-
ates a sequence of linearly independent vectors $M(t)$, that
is, a sequence of linearly independent variations per unit
stepsize $A(t)$, $B(t)$, C.

7.   Conjugate Gradient Phase: Practical Implementation

In this section, we summarize the results of Sections 5-6 and suggest practical ways of utilizing these results, within the following operating rules: (i) the use of second derivatives must be avoided; and (ii) the simultaneous determination of the directional coefficient γ and the stepsize α must be avoided.   We refer to the general case where the functional (1) is nonquadratic and/or the constraints (2)-(6) are nonlinear.

7.1. Auxiliary Functions.   The first step is to solve Eqs. (51)-(60) for a fictitious value of the directional coefficient, namely,

$$\gamma_* = 0 . \tag{94}$$

Using the solution technique of Section 4.4, we obtain the following auxiliary functions and multipliers:

$$A_*(t),\ B_*(t),\ C_*, \lambda_*(t),\ \rho_*(t),\ \sigma_*,\ \mu_* . \tag{95}$$

7.2. Directional Coefficient.   The second step is to compute the actual value of the directional coefficient γ .   For the first conjugate gradient phase, we set

$$\gamma = 0 . \tag{96}$$

For subsequent conjugate gradient phases, we set

$$\gamma = Q/\hat{Q} , \tag{97}$$

where

$$Q = \int_0^1 B_\star^T B_\star dt + C_\star^T C_\star + (E_\star^T E_\star)_0 , \tag{98-1}$$

$$\hat{Q} = \int_0^1 \hat{B}_\star^T \hat{B}_\star dt + \hat{C}_\star^T \hat{C}_\star + (\hat{E}_\star^T \hat{E}_\star)_0 . \tag{98-2}$$

In Eqs. (97)-(98), the symbols Q and $\hat{Q}$ denote the errors in the optimality conditions for the present conjugate gradient phase and the previous conjugate gradient phase, respectively.

The directional coefficient (97) is acceptable only if

$$\tilde{J}_\alpha(0) = -(Q + \gamma Z) < 0 , \tag{99}$$

where Q is given by (98-1) and Z is given by

$$Z = \int_0^1 B_\star^T \hat{B} dt + C_\star^T \hat{C} + (E_\star^T \hat{E})_0 . \tag{100}$$

If Ineq. (99) is violated, then the directional coefficient (97) must be discarded and replaced by the value (96). This means that the algorithm must be restarted by replacing the conjugate gradient phase with an ordinary gradient phase.

7.3. Basic Functions. The third step is to compute the basic functions $A(t)$, $B(t)$, C and the multipliers $\lambda(t)$, $\rho(t)$, σ, μ. This is done with the following formulas:

$$A(t) = A_*(t) + \gamma\hat{A}(t), \quad B(t) = B_*(t) + \gamma\hat{B}(t), \quad C = C_* + \gamma\hat{C}, \quad (101)$$

$$\lambda(t) = \lambda_*(t), \quad \rho(t) = \rho_*(t), \quad \sigma = \sigma_*, \quad \mu = \mu_*. \quad (102)$$

Therefore, in the practical implementation of the algorithm, the basic functions $A(t)$, $B(t)$, C are computed using the formulas derived under the assumption of linear constraints.

7.4. <u>Stepsize</u>. With the basic functions (101) known, we consider the one-parameter family of varied functions

$$\tilde{x}(t) = x(t) + \alpha A(t), \quad \tilde{u}(t) = u(t) + \alpha B(t), \quad \tilde{\pi} = \pi + \alpha C. \quad (103)$$

After substitution of Eqs. (102)-(103) into (8) and (15), the following functions of the stepsize are obtained:

$$\tilde{J} = \tilde{J}(\alpha), \quad \tilde{P} = \tilde{P}(\alpha). \quad (104)$$

Then, a one dimensional search scheme is applied to (104-1), and a value of the stepsize α is selected for which the following relations are satisfied:

$$\tilde{J}(\alpha) < \tilde{J}(0), \quad \tilde{P}(\alpha) \le P_*, \quad \tilde{\tau}(\alpha) > 0, \quad (105)$$

where τ is the final time and P_* is a preselected number, not necessarily small. Satisfaction of Ineq. (105-1) is possible because of the descent property of the conjugate gradient phase. Ineq. (105-2) is introduced to prevent excessive constraint violation. And Ineq. (105-3) is required

for problems with free final time.

Prior to the satisfaction of (105), a scanning process
is employed, leading to the bracketing of the minimum point
for $\tilde{J}(\alpha)$. This operation is then followed by a Hermitian
cubic interpolation process (Ref. 9), which is stopped whenever
the following relation is satisfied:[13]

$$|\tilde{J}_\alpha(\alpha)| \leq \varepsilon_3 \qquad \text{or} \qquad |\tilde{J}_\alpha(\alpha)/\tilde{J}_\alpha(0)| \leq \varepsilon_4, \qquad (106)$$

subject to an upper limit for the number of search steps N_s.
Once a stepsize α_o has been selected consistently with either
(106) or the prescribed upper limit for the number of search
steps, Ineqs. (105) must be checked. If satisfaction occurs,
then the stepsize α_o is accepted. If any violation occurs,
then the stepsize α_o must be bisected progressively until
satisfaction of (105) is finally achieved.

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[13]The symbols ε_3 and ε_4 denote small, preselected numbers.

8.   Descent Property of a Cycle

A descent property exists for a complete conjugate gradient-restoration cycle under the assumption of small stepsizes. Let α_g denote the conjugate gradient stepsize and α_r the restoration stepsize. Simple manipulations, omitted for the sake of brevity, show that the conjugate gradient corrections are of $O(\alpha_g)$, while the restoration corrections are of $O(\alpha_r \alpha_g^2)$. Hence, for α_g sufficiently small, the restoration corrections are negligible with respect to the conjugate gradient corrections. Therefore, the restoration phase preserves the descent property of the conjugate gradient phase.

More specifically, let I_1, I_2, I_3 denote the values of the functional (1) at the beginning of the conjugate gradient phase, at the end of the conjugate gradient phase, and at the end of the subsequent restoration phase. Note that I_1 and I_2 are not comparable, since the constraints are not satisfied to the same accuracy. On the other hand, I_1 and I_3 are comparable, and the conjugate gradient stepsize α_g can be selected so that

$$I_3 < I_1 . \tag{107}$$

This inequality constitutes the descent property of a complete conjugate gradient-restoration cycle. In order to enforce it, one proceeds as follows. At the end of the restoration phase,

one must verify Ineq. (107). If it is satisfied, the next

conjugate gradient phase is started; otherwise, the previous

conjugate gradient stepsize is bisected as many times as

needed until, after restoration, Ineq. (107) is satisfied.

9.   Summary of the Algorithm

The sequential conjugate gradient-restoration algorithm
solves optimal control problems involving a functional, sub-
ject to differential constraints, nondifferential constraints,
and general boundary conditions. The algorithm is composed
of a sequence of cycles, each cycle consisting of two phases,
a conjugate gradient phase and a restoration phase. The
objective of each cycle is to decrease the functional I,
while the constraints are satisfied to the predetermined
accuracy (18-1).

The decision parameters controlling the algorithm are
the constraint error P and the optimality condition error Q
[see Eqs. (15) and (16)]. If P violates Ineq. (18-1), the
algorithm executes a restoration phase. If P satisfies
Ineq. (18-1) and Q violates Ineq. (18-2), the algorithm executes a
conjugate gradient phase. Finally, if P and Q satisfy Ineqs.
(18), the algorithm stops: convergence has been achieved.

9.1. Restoration Phase. This phase involves one or
more iterations and can be summarized as follows.

(a) Assume nominal functions $x(t)$, $u(t)$, π which satis-
fy condition (4), but violate at least one of conditions
(2)-(3) and (5)-(6).

(b) For the nominal functions, solve the LTP-BVP
(32)-(36) and (38)-(42) using the method of particular

solutions.   In this way, obtain the functions A(t), B(t), C(t)
and the multipliers $\lambda(t)$, $\rho(t)$, σ , μ .

(c)   Using the functions in (b), compute the restoration
stepsize by a one-dimensional search on the constraint error
$\tilde{P}(\alpha)$.   To this effect, perform a bisection process on α,
starting from $\alpha_O = 1$, until Ineqs. (50) are satisfied.

(d)   Once the restoration stepsize is known, compute the
varied functions $\tilde{x}(t)$, $\tilde{u}(t)$, $\tilde{\pi}$ with Eqs. (23).

(e)   Verify whether the varied functions in (d) satisfy
Ineq. (18-1).   If this is the case, the restoration phase is
terminated.   Otherwise, return to (a) and continue the pro-
cess until satisfaction of (18-1) occurs.

9.2.   Conjugate Gradient Phase.   This phase involves a
single iteration and can be summarized as follows.

(a)   Assume nominal functions x(t), u(t), π which satisfy
the constraints (2)-(6) within the preselected accuracy (18-1).

(b)   For the nominal functions and for $\gamma_* = 0$, solve the
LTP-BVP (51)-(60) using the method of particular solutions.
In this way, obtain the auxiliary functions $A_*(t)$, $B_*(t)$, C_*
and the multipliers $\lambda_*(t)$, $\rho_*(t)$, σ_*, μ_* .

(c)   Set the directional coefficient γ at the level (96)
for the first conjugate gradient phase and at the level (97)
for any subsequent conjugate gradient phase.   In the latter
case, accept the directional coefficient only if Ineq. (99)

is satisfied; otherwise, reset γ at the level (96).

(d)   Compute the basic functions $A(t)$, $B(t)$, C and the multipliers $\lambda(t)$, $\rho(t)$, σ, μ using Eqs. (101)-(102).

(e)   Using the functions in (d), compute the conjugate gradient stepsize by a one-dimensional search on the augmented functional $\tilde{J}(\alpha)$ until satisfaction of Ineq. (106) occurs. Then, bisect the resulting stepsize α_o (if necessary), until satisfaction of Ineqs. (105) occurs.

(f)   Once the conjugate gradient stepsize is known, compute the varied functions $\tilde{x}(t)$, $\tilde{u}(t)$, $\tilde{\pi}$ with Eqs. (103).

9.3.   <u>Conjugate Gradient-Restoration Cycle</u>.   After the restoration phase is completed, verify whether Ineq. (107) is satisfied.   If this is the case, start the next cycle of the sequential conjugate gradient-restoration algorithm.   If not, return to the previous conjugate gradient phase and reduce the conjugate gradient stepsize (using a bisection process) until, after restoration, Ineq. (107) is satisfied.

10.   Experimental Conditions

In order to evaluate the theory, twelve examples were
solved. The sequential conjugate gradient-restoration
algorithm was programmed in FORTRAN IV, and the numerical
results were obtained in double-precision arithmetic.

Computations were performed at Rice University using an
IBM 370/155 computer. For each example, the interval of in-
tegration was divided into 100 steps. The differential equa-
tions were integrated using Hamming's modified predictor-
corrector method with a special Runge-Kutta starting procedure
(Ref. 10). The definite·integrals I, J, P, Q were computed
using a modified Simpson's rule. The method of particular
solutions (Refs. 6-8) was used to solve the linear, two-point
boundary-value problems associated with both the conjugate
gradient phase and the restoration phase.

10.1. Convergence Conditions. The parameters ε_1, ε_2,
ε_4 appearing in Ineqs. (18) and (106) were set at the levels[14]

$$\varepsilon_1 = E - 08, \qquad \varepsilon_2 = E - 04, \qquad \varepsilon_4 = E - 03. \qquad (108)$$

The tolerance level (108-1) characterizes the restoration
phase; the tolerance levels (108-1) and (108-2), employed in

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[14]The symbol $E \pm ab$ stands for $10^{\pm ab}$.

combination, characterize the algorithm as a whole; and the
tolerance level (108-3) characterizes the one-dimensional
search for the conjugate gradient stepsize.

10.2. Safeguards. For the conjugate gradient phase,
the parameter P_* appearing in Ineq. (105-2) was set at the
level

$$P_* = 10. \tag{109}$$

The tolerance level (109) limits the constraint violation
which is permissible during the conjugate gradient phase.
Also for the conjugate gradient phase, the number of Hermitian
search steps required to satisfy Ineq. (106) was subject to
the upper bound

$$N_s \leq 10. \tag{110}$$

10.3. Nonconvergence Conditions. The sequential conju-
gate gradient-restoration algorithm was programmed to stop
whenever satisfaction of any of the following inequalities
occurred:

$$\text{(i)} \quad N > 50 , \tag{111}$$

$$\text{(ii)} \quad N_c > 30 , \tag{112}$$

$$\text{(iii)} \quad N_r > 10 , \tag{113}$$

$$\text{(iv)} \qquad N_{br} > 10, \qquad\qquad\qquad\qquad (114)$$

$$\text{(v)} \qquad \tilde{J}_\alpha(0) \geq 0, \qquad \gamma = 0, \qquad\qquad (115)$$

$$\text{(vi)} \qquad N_{bg} > 10, \qquad \gamma = 0, \qquad\qquad (116)$$

$$\text{(vii)} \qquad N_{bc} > 10, \qquad \gamma = 0. \qquad\qquad (117)$$

Here, N is the total number of iterations, N_c is the number of cycles, N_r is the number of restorative iterations per cycle, N_{br} is the number of bisections of the restoration stepsize required to satisfy Ineqs. (50), N_{bg} is the number of bisections of the conjugate gradient stepsize required to satisfy Ineqs. (105), N_{bc} is the number of bisections of the conjugate gradient stepsize required to satisfy Ineq. (107), and $\tilde{J}_\alpha(0)$ is the slope of the augmented functional at $\alpha = 0$.

Inequalities (111)-(112) apply to the algorithm as a whole. Satisfaction of (111) and/or (112) is indicative of extreme slowness of convergence.

Inequalities (113)-(114) apply to the restoration phase. Satisfaction of (113) is indicative of failure to produce a feasible solution in a reasonable number of restorative iterations. Satisfaction of (114) is indicative of extreme smallness of the restorative displacements.

Inequalities (115)-(116) apply to the conjugate gradient

phase. Satisfaction of (115) means that the descent property

of the conjugate gradient phase does not hold, owing to

numerical inaccuracy. Satisfaction of (116) is indicative of

extreme smallness of the conjugate gradient displacements.

Inequality (117) applies to a complete conjugate

gradient-restoration cycle. Satisfaction of (117) is indica-

tive of extreme smallness of the displacements produced within

a complete conjugate gradient-restoration cycle.

10.4. <u>Restarting Conditions</u>. The directional coeffi-

cient γ of the <u>present</u> conjugate gradient phase was reset at

the level $\gamma = 0$ whenever satisfaction of any of the following

inequalities occurred:

$$(i) \qquad \tilde{J}_\alpha(0) \geq 0 , \qquad \gamma = Q/\hat{Q} , \qquad (118)$$

$$(ii) \qquad N_{bg} > 10 , \qquad \gamma = Q/\hat{Q} , \qquad (119)$$

$$(iii) \qquad N_{bc} > 10 , \qquad \gamma = Q/\hat{Q} . \qquad (120)$$

Satisfaction of (118) means that the descent property of

the conjugate gradient phase does not hold. Satisfaction of

(119) is indicative of extreme smallness of the conjugate

gradient displacements. And satisfaction of (120) is indica-

tive of extreme smallness of the displacements produced within

a complete conjugate gradient-restoration cycle.

The directional coefficient γ of the <u>next</u> conjugate gradient phase was reset at the level $\gamma = 0$ whenever satisfaction of any of the following inequalities occurred:

(iv) $\qquad 1 \le N_{bg} \le 10$, $\qquad \gamma = 0 \quad$ or $\quad \gamma = Q/\hat{Q}$, $\qquad$ (121)

(v) $\qquad 1 \le N_{bc} \le 10$, $\qquad \gamma = 0 \quad$ or $\quad \gamma = Q/\hat{Q}$. $\qquad$ (122)

Satisfaction of (121) or (122) is indicative of large violations of the orthogonality and conjugacy conditions, owing to the fact that the optimal conjugate gradient stepsize cannot be employed.

11.   Numerical Examples

In this section, twelve numerical examples are des-
cribed employing scalar notation. In particular, the symbols
$x_i(t)$, $i=1,\ldots,n$, denote the components of the state; the
symbols $u_i(t)$, $i=1,\ldots,m$, denote the components of the control;
and the symbols π_i, $i=1,\ldots,p$, denote the components of the
parameter.

For all of the examples, a time normalization is used in
order to simplify the numerical computations. Specifically,
the actual time θ is replaced by the normalized time

$$t = \theta/\tau, \tag{123}$$

which is defined in such a way that $t = 0$ at the initial point
and $t = 1$ at the final point. The actual final time τ, if it
is free, is regarded as a component of the vector parameter π
to be optimized. In this way, an optimal control problem with
variable final time is converted into an optimal control prob-
lem with fixed final time.

Example 11.1. This is a problem with (i) free initial
state and (ii) fixed final time $\tau = 1$:

$$I = \int_0^1 (x_1^2 + x_2^2 + u_1^2 + u_2^2)\,dt, \tag{124}$$

$$\dot{x}_1 = x_2, \qquad \dot{x}_2 = u_1 - x_2, \tag{125}$$

$$u_1 - u_2 + 5t - 3 = 0 , \tag{126}$$

$$x_1(1) = 0 . \tag{127}$$

The assumed nominal functions are:

$$x_1(t) = 0, \qquad x_2(t) = -1, \qquad u_1(t) = 0, \qquad u_2(t) = 0. \tag{128}$$

The numerical results are given in Tables 1-2. Convergence to the desired stopping condition occurs in $N = 3$ iterations, which include 1 restorative iteration and 2 conjugate gradient iterations.

Example 11.2. This is a problem with (i) a linear relation between the components of the initial state and (ii) fixed final time $\tau = 1$:

$$I = \int_0^1 (x_1^2 + x_2^2 + u_1^2/200 + u_2^2)\,dt, \tag{129}$$

$$\dot{x}_1 = x_2 , \qquad \dot{x}_2 = u_1 + u_2 - x_2 , \tag{130}$$

$$u_1 + 2u_2 - 10/(1 + 10t)^2 = 0, \tag{131}$$

$$x_1(0) + x_2(0) = -1, \tag{132}$$

$$x_1(1) + x_2(1) = 1 . \tag{133}$$

The assumed nominal functions are:

$$x_1(t) = 0, \qquad x_2(t) = 2t - 1, \qquad u_1(t) = 0, \qquad u_2(t) = 0. \qquad (134)$$

The numerical results are given in Tables 3-4. Convergence to the desired stopping condition occurs in $N = 3$ iterations, which include 1 restorative iteration and 2 conjugate gradient iterations.

Example 11.3. This is a problem with (i) initial state given and (ii) fixed final time $\tau = 1$:

$$I = \int_0^1 (x_1^2 + u_1^2)\, dt , \qquad (135)$$

$$\dot{x}_1 = x_1^2 - u_1, \qquad \dot{x}_2 = u_2 , \qquad (136)$$

$$x_1^2 - u_1 - 2x_2 u_2 = 0, \qquad (137)$$

$$x_1(0) = 1, \qquad x_2(0) = \sqrt{(0.1)}, \qquad (138)$$

$$x_1(1) = 1 . \qquad (139)$$

This problem is equivalent (Ref. 2) to that of minimizing (135), subject to (136-1), (138-1), (139), and the following state inequality constraint:

$$x_1 - 0.9 \geq 0 . \qquad (140)$$

The assumed nominal functions are:

$$x_1(t) = 1, \qquad x_2(t) = \sqrt{(0.1)}, \qquad u_1(t) = 1, \qquad u_2(t) = 1. \qquad (141)$$

The numerical results are given in Tables 5-6. Convergence to the desired stopping condition occurs in $N = 14$ iterations, which include 9 restorative iterations and 5 conjugate gradient iterations.

Example 11.4. This is a problem with (i) initial state partially given and (ii) fixed final time $\tau = 1$:

$$I = \int_0^1 \left[2x_2 u_1^3 / (1 + u_1^2) \right] dt + x_2^2(0), \qquad (142)$$

$$\dot{x}_1 = x_2^2, \qquad\qquad \dot{x}_2 = u_1, \qquad (143)$$

$$\dot{u}_1 - u_2^2 = 0, \qquad (144)$$

$$x_1(0) = 0, \qquad (145)$$

$$x_1(1) = 1/3 . \qquad (146)$$

This problem is equivalent (Ref. 2) to that of minimizing (142), subject to (143), (145), (146), and the following control inequality constraint:

$$u_1 \geq 0 . \qquad (147)$$

The assumed nominal functions are:[15]

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[15] Here, $C = \sqrt{(1/7)}$. For this value of C, the constraints (143)-(146) are satisfied.

$$x_1(t) = (C^2/3)(t^3 + 3t^2 + 3t), \qquad x_2(t) = C(t+1), \qquad (148)$$

$$u_1(t) = C, \qquad u_2(t) = \sqrt{C}. \qquad (149)$$

The numerical results are given in Tables 7-8. Convergence to the desired stopping condition occurs in $N = 7$ iterations, which include 4 restorative iterations and 3 conjugate gradient iterations.

Example 11.5. This is a minimum time problem with (i) a linear relation between the components of the initial state and (ii) free final time τ. After setting $\pi_1 = \tau$, the problem is as follows:

$$I = \pi_1 , \qquad (150)$$

$$\dot{x}_1 = \pi_1 u_1 , \qquad \dot{x}_2 = \pi_1 (u_1^2 - x_1^2 - 1/2) , \qquad (151)$$

$$u_1^2 - x_1^2 - u_2^2 = 0 , \qquad (152)$$

$$x_1(0) - x_2(0) = 0 , \qquad (153)$$

$$x_1(1) = 1 , \qquad x_2(1) = -\pi/4 . \qquad (154)$$

This problem is equivalent (Ref. 2) to that of minimizing (150), subject to (151), (153), (154), and the following state-derivative inequality constraint:

$$\dot{x}_2/\pi_1 + 1/2 \geq 0 . \tag{155}$$

The assumed nominal functions are:

$$x_1(t) = t, \qquad x_2(t) = -(\pi/4)t, \tag{156}$$

$$u_1(t) = 1, \qquad u_2(t) = 1, \qquad \pi_1 = 1. \tag{157}$$

The numerical results are given in Tables 9-10. Convergence to the desired stopping condition occurs in $N = 18$ iterations, which include 13 restorative iterations and 5 conjugate gradient iterations.

Example 11.6. This is a problem with (i) a component of the initial state given, (ii) a nonlinear relation concerning the remaining component of the initial state, and (iii) fixed final time $\tau = 1$:

$$I = \int_0^1 \left[2x_2 u_1^3 / (1+u_1^2) \right] dt + x_2^2(0) , \tag{158}$$

$$\dot{x}_1 = x_2^2 , \qquad \dot{x}_2 = u_1 , \tag{159}$$

$$u_1 - u_2^2 = 0, \tag{160}$$

$$x_1(0) = 0, \qquad \left[x_2(0) - 0.35 \right] \left[1 - x_2(0) \right] - \pi_1^2 = 0, \tag{161}$$

$$x_1(1) = 1/2 . \tag{162}$$

This problem is equivalent (Ref. 2) to that of minimizing (158), subject to (159), (161-1), (162), and the following inequality constraints imposed on the control and the initial state:

$$u_1 \geq 0, \tag{163}$$

$$0.35 \leq x_2(0) \leq 1. \tag{164}$$

The assumed nominal functions are:[16]

$$x_1(t) = (t^3 + 3t^2 + 3t)/14, \qquad x_2(t) = C(t+1), \tag{165}$$

$$u_1(t) = C, \qquad u_2(t) = \sqrt{C}, \qquad \pi_1 = \sqrt{[(C-0.35)(1-C)]}. \tag{166}$$

The numerical results are given in Tables 11-12. Convergence to the desired stopping condition occurs in $N = 10$ iterations, which include 7 restorative iterations and 3 conjugate gradient iterations.

Example 11.7. This is a problem with (i) a nonlinear relation between the components of the initial state and (ii) fixed final time $\tau = 1$:

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[16]Here, $C = \sqrt{(1/7)}$.

$$I = \int_0^1 (x_1^2 + u_1^2)\,dt, \tag{167}$$

$$\dot{x}_1 = x_1^2 - u_1, \qquad \dot{x}_2 = u_2, \tag{168}$$

$$x_1^2 - u_1 - 1 + 2t - 2x_2 u_2 = 0, \tag{169}$$

$$x_1(0) - x_2^2(0) = 0.8, \tag{170}$$

$$x_1(1) = 1. \tag{171}$$

This problem is equivalent (Ref. 2) to that of minimizing (167), subject to (168-1)', (171), and the following state inequality constraint:

$$x_1 - 0.8 - t + t^2 \geq 0. \tag{172}$$

The assumed nominal functions are:

$$x_1(t) = 1, \qquad x(t) = \sqrt{(0.2)}, \qquad u_1(t) = 1, \qquad u_2(t) = 0. \tag{173}$$

The numerical results are given in Tables 13-14. Convergence to the desired stopping condition occurs in $N = 22$ iterations, which include 13 restorative iterations and 9 conjugate gradient iterations.

Example 11.8. This is a problem with (i) a component of the initial state given, (ii) a linear relation between the remaining components of the initial state, and (iii) fixed

final time $\tau = \pi/2$:

$$I = \int_0^1 \tau(u_1^2 - x_1^2 + \tau t)dt, \tag{174}$$

$$\dot{x}_1 = \tau u_1, \quad \dot{x}_2 = \tau(2-4x_1^2), \quad \dot{x}_3 = \tau x_4, \quad \dot{x}_4 = \tau u_2, \tag{175}$$

$$4x_1 u_1 - x_3 u_2 - x_4^2 = 0, \tag{176}$$

$$x_1(0) = 0, \qquad 2x_3(0) + x_4(0) = 1, \tag{177}$$

$$x_1(1) = 1, \qquad x_2(1) = 0. \tag{178}$$

The assumed nominal functions are:

$$x_1(t) = t, \qquad x_2(t) = 4t(1-t), \qquad x_3(t) = 1 - 2t, \tag{179}$$

$$x_4(t) = -1, \qquad u_1(t) = 1/\tau, \qquad u_2(t) = 0. \tag{180}$$

The numerical results are given in Tables 15-16. Convergence to the desired stopping condition occurs in $N = 23$ iterations, which include 13 restorative iterations and 10 conjugate gradient iterations.

Example 11.9. This is a problem with (i) a component of the initial state given, (ii) a nonlinear relation between the remaining components of the initial state, and (iii) fixed final time $\tau = \pi/2$:

$$I = \int_0^1 \tau (u_1^2 - x_1^2 + \tau t) dt, \tag{181}$$

$$\dot{x}_1 = \tau u_1, \qquad \dot{x}_2 = \tau (2 - 4x_1^2), \qquad \dot{x}_3 = \tau x_4, \qquad \dot{x}_4 = \tau u_2, \tag{182}$$

$$4x_1 u_1 - x_3 u_2 - x_4^2 = 0, \tag{183}$$

$$x_1(0) = 0, \qquad x_2(0) + x_3(0) \left[x_3(0) + 2x_4(0) \right] = -1, \tag{184}$$

$$x_1(1) = 1, \qquad x_2(1) = 0. \tag{185}$$

The assumed nominal functions are:

$$x_1(t) = t, \qquad x_2(t) = 4t(1-t), \qquad x_3(t) = 1-2t, \qquad x_4(t) = -1, \tag{186}$$

$$u_1(t) = 1/\tau, \qquad u_2(t) = 0. \tag{187}$$

The numerical results are given in Tables 17-18. Convergence to the desired stopping condition occurs in $N = 20$ iterations, which include 12 restorative iterations and 8 conjugate gradient iterations.

Example 11.10. This is a problem with (i) a component of the initial state given, (ii) two nonlinear relations between the remaining components of the initial state, and (iii) fixed final time $\tau = \pi/2$:

$$I = \int_0^1 \tau (u_1^2 - x_1^2 + \tau t)\,dt, \tag{188}$$

$$\dot{x}_1 = \tau u_1, \qquad \dot{x}_2 = \tau(2 - 4x_1^2), \qquad \dot{x}_3 = \tau x_4, \qquad \dot{x}_4 = \tau u_2, \tag{189}$$

$$4x_1 u_1 - x_3 u_2 - x_4^2 = 0, \tag{190}$$

$$x_1(0) = 0, \qquad x_2(0) + x_3^2(0) = 1, \qquad x_3(0)x_4(0) = -1, \tag{191}$$

$$x_1(1) = 1, \qquad x_2(1) = 0. \tag{192}$$

This problem is equivalent (Ref. 2) to that of minimizing (188), subject to (189-1), (189-2), (191-1), (192), and the following state inequality constraint:

$$1 - x_2 \geq 0. \tag{193}$$

The assumed nominal functions are:

$$x_1(t) = t, \qquad x_2(t) = 4t(1-t), \qquad x_3(t) = 1-2t, \qquad x_4(t) = -1, \tag{194}$$

$$u(t) = 1/\tau, \qquad u_2(t) = 0. \tag{195}$$

The numerical results are given in Tables 19-20. Convergence to the desired stopping condition occurs in $N = 18$ iterations, which include 11 restorative iterations and 7 conjugate gradient iterations.

Example 11.11. This is a minimum time problem with (i)

a nonlinear relation between the components of the initial state and (ii) free final time τ. After setting $\pi_1 = \tau$, the problem is as follows:

$$I = \pi_1 , \tag{196}$$

$$\dot{x}_1 = \pi_1 u_1 , \qquad \dot{x}_2 = \pi_1 (u_1^2 - x_1^2 - 1/2) , \tag{197}$$

$$u_1^2 - x_1^2 - u_2^2 = 0 , \tag{198}$$

$$x_1(0) + x_2^2(0) = 0 , \tag{199}$$

$$\dot{x}_1(1) x_2(1) = -\pi/4 . \tag{200}$$

This problem is equivalent (Ref. 2) to that of minimizing (196), subject to (197), (199), (200), and the following state-derivative inequality constraint:

$$\dot{x}_2/\pi_1 + 1/2 \geq 0 . \tag{201}$$

The assumed nominal functions are:

$$x_1(t) = t, \quad x_2(t) = -(\pi/4)t, \quad u_1(t) = 1, \quad u_2(t) = 1, \quad \pi_1 = 1. \tag{202}$$

The numerical results are given in Tables 21-22. Convergence to the desired stopping condition occurs in $N = 14$ iterations, which include 10 restorative iterations and 4 conjugate gradient iterations.

Example 11.12.   This is a problem with (i) a linear relation between the first two components of the initial state, (ii) a nonlinear relation between the remaining components of the initial state, and (ii) fixed final time $\tau = 1$:

$$I = \int_0^1 u_1^2 dt, \qquad (203)$$

$$\dot{x}_1 = x_2, \qquad \dot{x}_2 = u_1, \qquad \dot{x}_3 = x_4, \qquad \dot{x}_4 = u_2, \qquad (204)$$

$$u_1 + 2x_3 u_2 + 2x_4^2 = 0, \qquad (205)$$

$$x_1(0) + x_2(0) = 1, \qquad x_3^2(0) + 2x_3(0)x_4(0) = -0.85, \qquad (206)$$

$$x_1(1) = 0, \qquad x_2(1) = -1. \qquad (207)$$

The assumed nominal functions are:

$$x_1(t) = 0, \qquad x_2(t) = 1 - 2t, \qquad x_3(t) = (1 - 2t)\sqrt{(0.15)}, \qquad (208)$$

$$x_4(t) = (2t-1)/2\sqrt{(0.15)}, \qquad u_1(t) = 1, \qquad u_2(t) = 0. \qquad (209)$$

The numerical results are given in Tables 23-24.   Convergence to the desired stopping condition occurs in $N = 20$ iterations, which include 12 restorative iterations and 8 conjugate gradient iterations.

Table 1. Convergence history, Example 11.1.

N_c	N_g	N_r	N	γ	P	Q	I
0	0	0	0		0.43E+01		
1	0	1	1		0.15E-29	0.17E+00	1.29984
2	1	0	2	$\gamma=0$	0.90E-30	0.49E-02	1.18165
3	1	0	3	$\gamma\neq0$	0.12E-29	0.31E-06	1.18040

Table 2. Converged solution, Example 11.1.

t	x_1	x_2	u_1	u_2
0.0	-0.0069	-0.3314	1.5004	-1.4995
0.1	-0.0316	-0.1699	1.2368	-1.2631
0.2	-0.0422	-0.0485	0.9798	-1.0201
0.3	-0.0425	0.0370	0.7279	-0.7720
0.4	-0.0359	0.0908	0.4799	-0.5200
0.5	-0.0253	0.1159	0.2343	-0.2656
0.6	-0.0135	0.1153	-0.0100	-0.0100
0.7	-0.0030	0.0916	-0.2544	0.2455
0.8	0.0040	0.0468	-0.5000	0.4999
0.9	0.0056	-0.0171	-0.7481	0.7518
1.0	0.0000	-0.0988	-1.0000	1.0000

$$\tau = 1.00000$$

Table 3. Convergence history, Example 11.2.

N_c	N_g	N_r	N	γ	P	Q	I
0	0	0	0		0.79E+01		
1	0	1	1		0.37E-27	0.35E+00	1.89493
2	1	0	2	$\gamma=0$	0.15E-26	0.65E-02	1.48434
3	1	0	3	$\gamma\neq0$	0.24E-26	0.50E-04	1.48042

Table 4. Converged solution, Example 11.2.

t	x_1	x_2	u_1	u_2
0.0	0.1127	-1.1127	12.0825	-1.0412
0.1	0.0416	-0.4340	4.6589	-1.0794
0.2	0.0146	-0.1335	3.2253	-1.0571
0.3	0.0114	0.0578	2.7069	-1.0409
0.4	0.0245	0.1984	2.4732	-1.0366
0.5	0.0501	0.3101	2.3668	-1.0445
0.6	0.0859	0.4035	2.3329	-1.0644
0.7	0.1304	0.4847	2.3482	-1.0959
0.8	0.1826	0.5581	2.4013	-1.1389
0.9	0.2419	0.6265	2.4864	-1.1932
1.0	0.3078	0.6921	2.5999	-1.2586

$$\tau = 1.00000$$

Table 5.   Convergence history, Example 11.3.

N_c	N_g	N_r	N	γ	P	Q	I
0	0	0	0		0.14E+01		
1	0	3	3		0.52E-09	0.35E+00	1.83569
2	1	2	6	$\gamma=0$	0.15E-16	0.14E-01	1.66599
3	1	1	8	$\gamma\neq0$	0.12E-08	0.27E-03	1.65745
4	1	1	10	$\gamma\neq0$	0.12E-12	0.18E-03	1.65704
5	1	1	12	$\gamma\neq0$	0.51E-13	0.16E-03	1.65677
6	1	1	14	$\gamma\neq0$	0.34E-12	0.66E-04	1.65641

Table 6.   Converged solution, Example 11.3.

t	x_1	x_2	u_1	u_2
0.0	1.0000	0.3162	1.7514	-1.1881
0.1	0.9417	0.2043	1.3437	-1.1180
0.2	0.9082	0.0907	1.0166	-1.0558
0.3	0.9002	0.0151	0.8232	-0.4215
0.4	0.9000	-0.0027	0.8099	-0.0180
0.5	0.9000	-0.0013	0.8100	0.0096
0.6	0.9000	-0.0010	0.8100	0.0303
0.7	0.9003	0.0175	0.7956	0.4250
0.8	0.9087	0.0933	0.6292	1.0524
0.9	0.9420	0.2051	0.4398	1.0912
1.0	1.0000	0.3162	0.2376	1.2053

$$\tau = 1.00000$$

Table 7.   Convergence history, Example 11.4.

N_c	N_g	N_r	N	γ	P	Q	I
0	0	0	0		0.47E-32	0.49E-01	0.19642
1	1	2	3	$\gamma=0$	0.10E-13	0.11E-02	0.16924
2	1	1	5	$\gamma\neq0$	0.12E-09	0.10E-03	0.16816
3	1	1	7	$\gamma\neq0$	0.31E-13	0.11E-04	0.16809

Table 8.   Converged solution, Example 11.4.

t	x_1	x_2	u_1	u_2
0.0	0.0000	0.2908	0.9570	0.9782
0.1	0.0112	0.3753	0.7474	0.8645
0.2	0.0281	0.4428	0.6127	0.7827
0.3	0.0504	0.4995	0.5266	0.7257
0.4	0.0779	0.5488	0.4603	0.6784
0.5	0.1105	0.5917	0.3978	0.6307
0.6	0.1478	0.6284	0.3372	0.5807
0.7	0.1894	0.6593	0.2817	0.5308
0.8	0.2346	0.6848	0.2269	0.4764
0.9	0.2829	0.7039	0.1491	0.3862
1.0	0.3333	0.7138	0.0516	0.2272

$$\tau = 1.00000$$

Table 9.   Convergence history, Example 11.5.

N_c	N_g	N_r	N	γ	P	Q	I
0	0	0	0		0.11E+01		
1	0	5	5		0.13E-14	0.21E-01	1.83056
2	1	2	8	$\gamma=0$	0.52E-14	0.37E-02	1.82421
3	1	2	11	$\gamma\neq0$	0.18E-15	0.20E-02	1.82301
4	1	2	14	$\gamma\neq0$	0.72E-15	0.11E-02	1.82249
5	1	1	16	$\gamma\neq0$	0.11E-08	0.27E-03	1.82230
6	1	1	18	$\gamma\neq0$	0.81E-11	0.71E-04	1.82226

Table 10.   Converged solution, Example 11.5.

t	x_1	x_2	u_1	u_2
0.0	0.0034	0.0034	0.4978	0.4978
0.1	0.0936	-0.0436	0.4887	0.4797
0.2	0.1805	-0.0967	0.4631	0.4264
0.3	0.2614	-0.1609	0.4225	0.3319
0.4	0.3338	-0.2396	0.3703	0.1601
0.5	0.4020	-0.3298	0.4025	-0.0193
0.6	0.4824	-0.4209	0.4824	0.0013
0.7	0.5788	-0.5120	0.5788	-0.0015
0.8	0.6945	-0.6031	0.6945	0.0027
0.9	0.8334	-0.6942	0.8334	-0.0029
1.0	1.0000	-0.7853	1.0000	-0.0016

$$\tau = \pi_1 = 1.82226$$

Table 11.   Convergence history, Example 11.6.

N_c	N_g	N_r	N	γ	P	Q	I
0	0	0	0		0.31E-01		
1	0	3	3		0.28E-09	0.65E-01	0.31197
2	1	2	6	$\gamma=0$	0.14E-15	0.17E-02	0.28910
3	1	1	8	$\gamma\neq0$	0.44E-10	0.27E-03	0.28781
4	1	1	10	$\gamma\neq0$	0.52E-12	0.19E-04	0.28762

Table 12.   Converged solution, Example 11.6.

t	x_1	x_2	u_1	u_2
0.0	0.0000	0.4133	0.9597	0.9796
0.1	0.0210	0.4990	0.7660	0.8752
0.2	0.0496	0.5686	0.6354	0.7971
0.3	0.0855	0.6275	0.5471	0.7396
0.4	0.1283	0.6787	0.4789	0.6920
0.5	0.1775	0.7235	0.4174	0.6460
0.6	0.2328	0.7623	0.3576	0.5980
0.7	0.2935	0.7950	0.2974	0.5454
0.8	0.3590	0.8216	0.2315	0.4812
0.9	0.4282	0.8409	0.1525	0.3906
1.0	0.5000	0.8518	0.0677	0.2602

$$\tau = 1.00000 \quad , \quad \pi_1 = 0.19278$$

Table 13.   Convergence history, Example 11.7.

N_c	N_g	N_r	N	γ	P	Q	I
0	0	0	0		0.33E+00		
1	0	3	3		0.52E-14	0.40E+01	2.57561
2	1	2	6	$\gamma=0$	0.32E-11	0.11E-01	1.82401
3	1	1	8	$\gamma\neq0$	0.25E-09	0.20E-01	1.81492
4	1	1	10	$\gamma\neq0$	0.18E-09	0.18E-02	1.80922
5	1	1	12	$\gamma\neq0$	0.11E-11	0.22E-02	1.80803
6	1	1	14	$\gamma\neq0$	0.31E-12	0.40E-03	1.80736
7	1	1	16	$\gamma\neq0$	0.37E-13	0.74E-03	1.80704
8	1	1	18	$\gamma\neq0$	0.35E-13	0.15E-03	1.80678
9	1	1	20	$\gamma\neq0$	0.15E-14	0.32E-03	1.80667
10	1	1	22	$\gamma\neq0$	0.14E-13	0.98E-04	1.80650

Table 14.   Converged solution, Example 11.7.

t	x_1	x_2	u_1	u_2
0.0	0.8139	0.1180	0.0102	-1.4724
0.1	0.8901	0.0124	0.0056	-0.5342
0.2	0.9600	-0.0063	0.3219	0.0235
0.3	1.0100	0.0000	0.6201	0.0633
0.4	1.0400	0.0033	0.8816	0.0012
0.5	1.0500	-0.0001	1.1024	-0.0725
0.6	1.0400	-0.0085	1.2808	-0.0525
0.7	1.0101	0.0108	1.4068	0.6212
0.8	0.9779	0.1340	1.1179	1.6351
0.9	0.9747	0.2910	0.8558	1.5361
1.0	1.0000	0.4472	0.6327	1.5286

$\tau = 1.00000$

Table 15. Convergence history, Example 11.8.

N_c	N_g	N_r	N	γ	P	Q	I
0	0	0	0		0.23E+01		
1	0	3	3		0.49E-09	0.52E-01	1.27196
2	1	1	5	$\gamma=0$	0.86E-09	0.27E-01	1.25174
3	1	1	7	$\gamma\neq0$	0.83E-09	0.10E-01	1.24337
4	1	1	9	$\gamma\neq0$	0.54E-10	0.45E-02	1.23945
5	1	1	11	$\gamma\neq0$	0.85E-11	0.32E-02	1.23734
6	1	1	13	$\gamma\neq0$	0.40E-11	0.22E-02	1.23589
7	1	1	15	$\gamma\neq0$	0.13E-11	0.13E-02	1.23492
8	1	1	17	$\gamma\neq0$	0.22E-12	0.79E-03	1.23435
9	1	1	19	$\gamma\neq0$	0.31E-13	0.43E-03	1.23403
10	1	1	21	$\gamma\neq0$	0.33E-14	0.21E-03	1.23385
11	1	1	23	$\gamma\neq0$	0.24E-15	0.96E-04	1.23377

Table 16. Converged solution, Example 11.8.

t	x_1	x_2	x_3	x_4	u_1	u_2
0.0	0.0000	-0.0122	0.9981	-0.9963	0.9955	-0.9944
0.1	0.1557	0.2968	0.8300	-1.1396	0.9715	-0.8354
0.2	0.3076	0.5758	0.6416	-1.2548	0.9557	-0.6208
0.3	0.4516	0.7978	0.4376	-1.3403	0.8747	-0.4941
0.4	0.5850	0.9417	0.2221	-1.3953	0.8130	-0.2003
0.5	0.7044	0.9927	0.0014	-1.4094	0.7050	0.0059
0.6	0.8062	0.9464	-0.2191	-1.3944	0.5899	0.1908
0.7	0.8890	0.8070	-0.4350	-1.3478	0.4611	0.4060
0.8	0.9502	0.5873	-0.6407	-1.2664	0.3157	0.6301
0.9	0.9876	0.3090	-0.8310	-1.1507	0.1589	0.8379
1.0	1.0000	0.0000	-1.0007	-1.0047	-0.0015	1.0148

$$\tau = \pi/2 = 1.57079$$

Table 17.   Convergence history, Example 11.9.

N_c	N_g	N_r	N	γ	P	Q	I
0	0	0	0		0.23E+01		
1	0	4	4		0.16E-13	0.22E-01	1.24364
2	1	1	6	$\gamma=0$	0.24E-10	0.55E-02	1.23682
3	1	1	8	$\gamma\neq0$	0.20E-11	0.13E-02	1.23518
4	1	1	10	$\gamma\neq0$	0.28E-13	0.54E-03	1.23465
5	1	1	12	$\gamma\neq0$	0.20E-14	0.37E-03	1.23438
6	1	1	14	$\gamma\neq0$	0.55E-15	0.28E-03	1.23419
7	1	1	16	$\gamma\neq0$	0.61E-15	0.20E-03	1.23403
8	1	1	18	$\gamma\neq0$	0.93E-15	0.12E-03	1.23394
9	1	1	20	$\gamma\neq0$	0.44E-15	0.72E-04	1.23388

Table 18.   Converged solution, Example 11.9.

t	x_1	x_2	x_3	x_4	u_1	u_2
0.0	0.0000	0.0422	1.0522	-1.0213	1.0068	-0.9914
0.1	0.1578	0.3511	0.8802	-1.1643	1.0051	-0.8188
0.2	0.3122	0.6292	0.6879	-1.2786	0.9674	-0.6198
0.3	0.4579	0.8486	0.4800	-1.3652	0.9013	-0.4433
0.4	0.5940	0.9872	0.2613	-1.4117	0.8199	-0.1709
0.5	0.7142	1.0306	0.0382	-1.4239	0.7099	0.0156
0.6	0.8166	0.9746	-0.1843	-1.4045	0.5904	0.2399
0.7	0.8985	0.8243	-0.4010	-1.3469	0.4496	0.4938
0.8	0.9570	0.5949	-0.6054	-1.2504	0.2931	0.7286
0.9	0.9906	0.3106	-0.7921	-1.1208	0.1347	0.9115
1.0	1.0000	0.0000	-0.9563	-0.9674	-0.0124	1.0307

$$\tau = \pi/2 = 1.57079$$

Table 19.   Convergence history, Example 11.10.

N_c	N_g	N_r	N	γ	P	Q	I
0	0	0	0		0.23E+01		
1	0	4	4		0.21E-15	0.24E-01	1.24401
2	1	1	6	$\gamma=0$	0.29E-10	0.60E-02	1.23670
3	1	1	8	$\gamma\neq0$	0.25E-11	0.15E-02	1.23495
4	1	1	10	$\gamma\neq0$	0.35E-13	0.60E-03	1.23438
5	1	1	12	$\gamma\neq0$	0.26E-14	0.40E-03	1.23408
6	1	1	14	$\gamma\neq0$	0.54E-15	0.24E-03	1.23389
7	1	1	16	$\gamma\neq0$	0.55E-16	0.12E-03	1.23378
8	1	1	18	$\gamma\neq0$	0.26E-17	0.48E-04	1.23374

Table 20.   Converged solution, Example 11.10.

t	x_1	x_2	x_3	x_4	u_1	u_2
0.0	0.0000	-0.0008	1.0004	-0.9995	1.0016	-0.9987
0.1	0.1563	0.3081	0.8317	-1.1434	0.9848	-0.8312
0.2	0.3087	0.5870	0.6426	-1.2594	0.9432	-0.6553
0.3	0.4534	0.8085	0.4374	-1.3456	0.9002	-0.4064
0.4	0.5878	0.9506	0.2221	-1.3909	0.8044	-0.1941
0.5	0.7061	0.9999	0.0018	-1.4090	0.7028	-0.0310
0.6	0.8081	0.9520	-0.2189	-1.3979	0.5922	0.1808
0.7	0.8910	0.8104	-0.4353	-1.3506	0.4601	0.4227
0.8	0.9517	0.5887	-0.6413	-1.2655	0.3100	0.6568
0.9	0.9880	0.3092	-0.8311	-1.1462	0.1527	0.8544
1.0	1.0000	0.0000	-1.0000	-0.9999	0.0007	0.9970

$$\tau = \pi/2 = 1.57079$$

Table 21.   Convergence history, Example 11.11.

N_c	N_g	N_r	N	γ	P	Q	I
0	0	0	0		0.11E+01		
1	0	4	4		0.28E-13	0.10E+00	1.61985
2	1	2	7	$\gamma=0$	0.39E-12	0.14E-01	1.58737
3	1	2	10	$\gamma\neq0$	0.99E-16	0.14E-02	1.58313
4	1	1	12	$\gamma\neq0$	0.19E-08	0.26E-03	1.58236
5	1	1	14	$\gamma\neq0$	0.10E-09	0.64E-04	1.58221

Table 22.   Converged solution, Example 11.11.

t	x_1	x_2	u_1	u_2
0.0	-0.0928	-0.3047	0.8187	0.8134
0.1	0.0372	-0.2771	0.8228	0.8219
0.2	0.1664	-0.2526	0.8066	0.7893
0.3	0.2914	-0.2414	0.7704	0.7131
0.4	0.4092	-0.2526	0.7144	0.5856
0.5	0.5165	-0.2930	0.6402	0.3781
0.6	0.6141	-0.3627	0.6184	0.0732
0.7	0.7194	-0.4417	0.7195	-0.0104
0.8	0.8428	-0.5208	0.8428	0.0051
0.9	0.9872	-0.5999	0.9873	-0.0034
1.0	1.1565	-0.6790	1.1565	-0.0087

$$\tau = \pi_1 = 1.58221$$

Table 23. Convergence history, Example 11.12.

N_c	N_g	N_r	N	γ	P	Q	I
0	0	0	0		0.22E+02		
1	0	4	4		0.17E-08	0.46E+00	3.32348
2	1	2	7	$\gamma=0$	0.67E-13	0.55E-01	3.05708
3	1	2	10	$\gamma\neq0$	0.96E-16	0.15E-01	3.01500
4	1	1	12	$\gamma=0$	0.11E-16	0.25E-02	3.00374
5	1	1	14	$\gamma=0$	0.20E-17	0.13E-02	3.00161
6	1	1	16	$\gamma=0$	0.17E-16	0.51E-03	3.00073
7	1	1	18	$\gamma=0$	0.81E-19	0.27E-03	3.00034
8	1	0	19	$\gamma=0$	0.61E-08	0.11E-03	3.00016
9	1	0	20	$\gamma=0$	0.63E-08	0.60E-04	3.00007

Table 24. Converged solution, Example 11.12.

t	x_1	x_2	x_3	x_4	u_1	u_2
0.0	0.5018	0.4981	0.5645	-1.0351	-3.0237	0.7798
0.1	0.5370	0.2121	0.4651	-0.9487	-2.7051	0.9725
0.2	0.5452	-0.0432	0.3756	-0.8348	-2.4024	1.3420
0.3	0.5294	-0.2684	0.2998	-0.6704	-2.1021	2.0061
0.4	0.4925	-0.4637	0.2443	-0.4231	-1.8023	2.9551
0.5	0.4377	-0.6282	0.2179	-0.0969	-1.4856	3.3646
0.6	0.3679	-0.7613	0.2240	0.2027	-1.1815	2.4533
0.7	0.2864	-0.8650	0.2543	0.3824	-0.8926	1.1796
0.8	0.1959	-0.9396	0.2968	0.4533	-0.5988	0.3164
0.9	0.0994	-0.9847	0.3428	0.4582	-0.3017	-0.1722
1.0	0.0000	-1.0000	0.3872	0.4253	-0.0033	-0.4628

$$\tau = 1.00000$$

12.   Discussion

   The examples presented in Section 11 were solved with
both the sequential conjugate gradient-restoration algorithm
(SCGRA) of this report and the sequential ordinary gradient-
restoration algorithm (SOGRA) of Ref. 5.   This was done in
order to gain perspective on the relative merit of SCGRA
vis-a-vis SOGRA.

   The comparative results[17] are presented in Tables 25-27,
where the number of iterations N required to achieve different
tolerance levels for the error in the optimality conditions Q
is given for a fixed tolerance level of the constraint error
$P \leq E-08$. Also shown in the tables are the values obtained
for the objective functional I.

   Cumulative results for the twelve examples investigated
are given in Table 28. Here, the total number of iterations
for convergence ΣN is presented as a function of the tolerance
level chosen for the error in the optimality conditions Q, for
a fixed tolerance level in the constraint error $P \leq E-08$. In
this comparative study, tolerance levels in the range $Q \leq E-02$
to $Q \leq E-04$ were chosen for the error in the optimality conditions.

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[17]In Tables 25-27, the symbol LQ stands for a linear-quadratic
problem, and the symbol NLQ stands for a nonlinear and/or
nonquadratic problem.

From Tables 25-28, it appears that there is no saving
in number of iterations for $Q \leq$ E-02. The saving is 4%
for $Q \leq$ E-03 and 11% for $Q \leq$ E-04. Clearly, the relative
advantage of SCGRA with respect to SOGRA increases by imposing
a tighter tolerance level on the error in the optimality con-
ditions.

It must be noted that the experiments performed show
that the computer time per iteration is roughly the same for
SCGRA and SOGRA. Therefore, the conclusions pertaining to
savings in number of iterations also apply to savings in com-
puter time.

Table 25.   Results for $P \leq$ E-08 and $Q \leq$ E-02.

Example	Type	SCGRA		SOGRA	
		N	I	N	I
11.1	LQ	2	1.18165	2	1.18165
11.2	LQ	2	1.48434	2	1.48434
11.3	NLQ	8	1.65745	8	1.65742
11.4	NLQ	3	0.16924	3	0.16924
11.5	NLQ	8	1.82421	8	1.82421
11.6	NLQ	6	0.28910	6	0.28910
11.7	NLQ	10	1.80922	10	1.81240
11.8	NLQ	9	1.23945	7	1.24498
11.9	NLQ	6	1.23682	6	1.23682
11.10	NLQ	6	1.23670	6	1.23670
11.11	NLQ	10	1.58313	9	1.58381
11.12	NLQ	12	3.00374	11	3.00785

Table 26.   Results for $P \leq$ E-08 and $Q \leq$ E-03.

Example	Type	SCGRA		SOGRA	
		N	I	N	I
11.1	LQ	3	1.18040	3	1.18041
11.2	LQ	3	1.48042	3	1.48045
11.3	NLQ	8	1.65745	8	1.65742
11.4	NLQ	5	0.16816	5	0.16818
11.5	NLQ	16	1.82230	12	1.82276
11.6	NLQ	8	0.28781	8	0.28774
11.7	NLQ	14	1.80736	18	1.80797
11.8	NLQ	17	1.23435	19	1.23613
11.9	NLQ	10	1.23465	10	1.23492
11.10	NLQ	10	1.23438	11	1.23439
11.11	NLQ	12	1.58236	13	1.58242
11.12	NLQ	16	3.00073	17	3.00075

Table 27.   Results for $P \leq E$-08 and $Q \leq E$-04.

Example	Type	SCGRA		SOGRA	
		N	I	N	I
11.1	LQ	3	1.18040	3	1.18041
11.2	LQ	3	1.48042	3	1.48045
11.3	NLQ	14	1.65641	12	1.65678
11.4	NLQ	7	0.16809	7	0.16810
11.5	NLQ	18	1.82226	20	1.82224
11.6	NLQ	10	0.28762	10	0.28764
11.7	NLQ	22	1.80650	26	1.80654
11.8	NLQ	23	1.23377	35	1.23400
11.9	NLQ	20	1.23388	20	1.23411
11.10	NLQ	18	1.23374	20	1.23384
11.11	NLQ	14	1.58221	16	1.58217
11.12	NLQ	20	3.00007	21	3.00007

Table 28.   Cumulative number of iterations
for convergence, $P \leq E$-08.

Q	SCGRA	SOGRA
	ΣN	ΣN
$Q \leq E$-02	82	78
$Q \leq E$-03	122	127
$Q \leq E$-04	172	193

13.   Conclusions

In this report, a new member of the family of sequential
gradient-restoration algorithms for the solution of optimal
control problems is presented.   This is an algorithm of the
conjugate gradient type and solves the problem represented by
Eqs. (1)-(6):   Minimize a functional subject to differential
constraints, nondifferential constraints, and general boundary
conditions.

The algorithm presented here differs from those of Refs.
3-4, in that it is not required that the state vector be given
at the initial point.   Instead, the initial conditions can be
absolutely general.   In analogy with Refs. 3-4, the present
algorithm is capable of handling general final conditions;
therefore, it is suitable for the solution of optimal control
problems with general boundary conditions.

The importance of the present algorithm lies in that many
optimal control problems either arise naturally in the present
format or can be brought to such a format by means of suitable
transformations (see Ref. 2).   Therefore, a great variety of
optimal control problems can be handled, as it is shown by the
numerical examples presented.

Twelve numerical examples are presented to illustrate the
performance of the algorithm.   The numerical results show the
feasibility as well as the convergence characteristics of the

algorithm. A comparative analysis of the sequential conjugate gradient-restoration algorithm and the sequential ordinary gradient-restoration algorithm shows that the relative advantage of SCGRA with respect to SOGRA increases by imposing a tighter tolerance level on the error in the optimality conditions.

In summary, the new member of the family of sequential gradient-restoration algorithms described here has the following properties: (i) it retains the robustness, reliability, and convergence characteristics of the algorithms discussed in Refs. 3-4; (ii) it is able to handle all of the optimal control problems treated in Refs. 3-4; and (iii) it has the additional capability of handling optimal control problems with general boundary conditions.

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